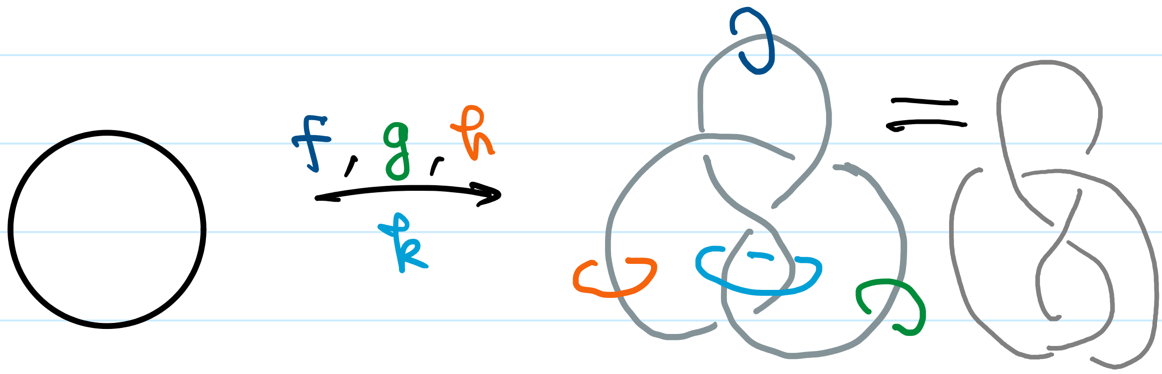
Lect22-20180411

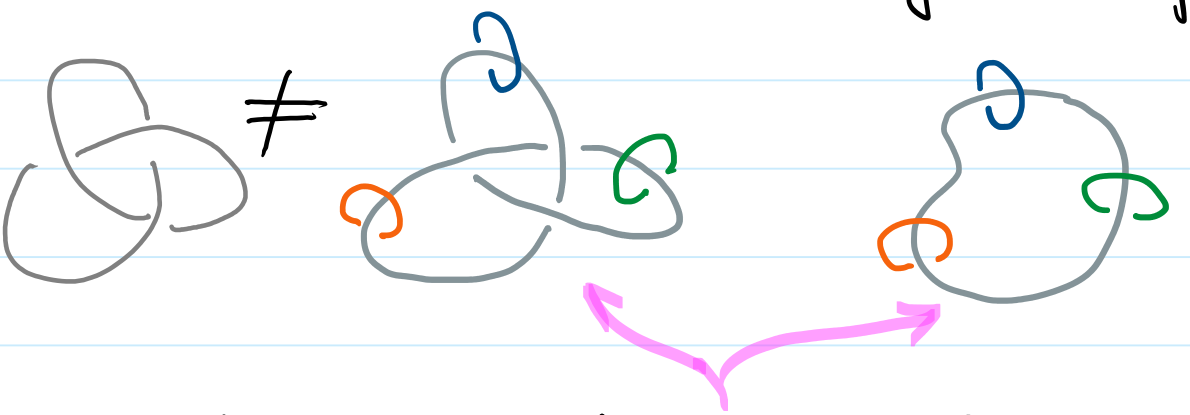
Monday, 9 April 2018 11:26 AM

Recall. Two continuous mappings $f, g : X \longrightarrow Y$ are homotopic (under $H : X \times [0,1] \longrightarrow Y$), denoted $f \overset{H}{\simeq} g$, if H is continuous and $\forall\ x \in X \quad H(x,0) = f(x)\ ,\ H(x,1) = g(x)$.

Example. Take $X = S^1$, $Y = \mathbb{R}^3 \setminus (\text{knot})$



Easy to expect $\underbrace{f \simeq g \simeq h}_{\text{This is not good enough}} \not\simeq k$

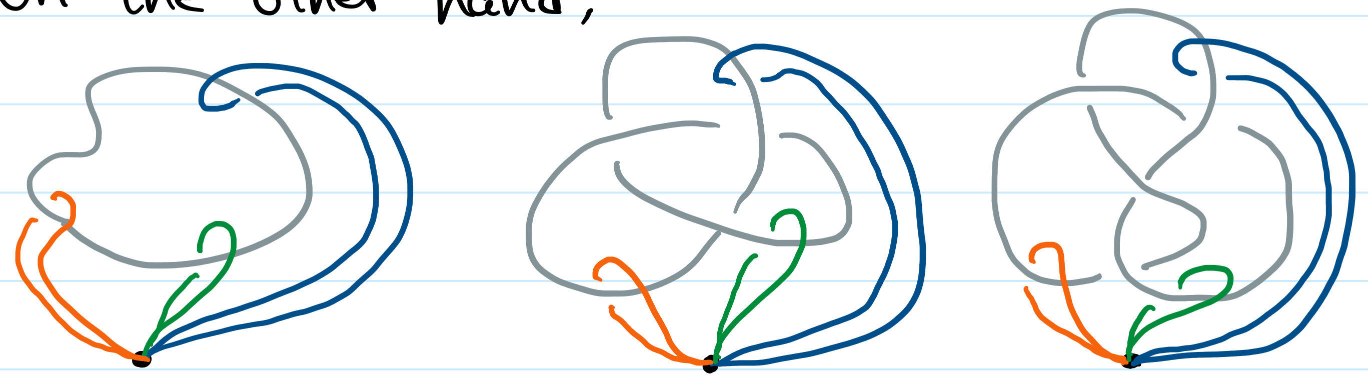


Similar situation occurs in every $\mathbb{R}^3 \setminus$ knot no matter how simple or complicated the knot is!

Lect22-p2

Apr 12, Thursday, 2018 4:22 PM

On the other hand,



$f \simeq g \simeq h$ $\qquad$ $f \not\simeq g \not\simeq h$ $\qquad$ $f \not\simeq g \not\simeq h$

In fact, the algebraic relation among them are different

Definition. Let $A \subset X$ and Y be spaces. Two continuous mappings $f, g : X \longrightarrow Y$ are homotopic rel A under $H : X \times [0,1] \longrightarrow Y$ if H is continuous such that

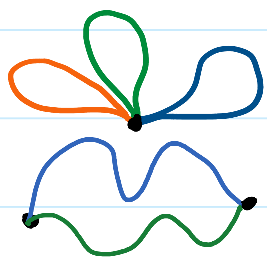
(1) $\forall\ x \in X$, $H(x,0) = f(x)$, $H(x,1) = g(x)$

(2) $\forall\ t \in [0,1]$, $\forall\ a \in A$ $H(a,t) = f(a) = g(a)$

i.e., $f|_A \equiv g|_A$ at first

In above, we study f, g, h in terms of

- $X = S^1$, $A = \{pt\}$ OR
- $X = [0,1]$, $A = \{0,1\}$



Definition. Let X be a topological space. Two paths $\gamma_0, \gamma_1 : [0,1] \longrightarrow X$ are **path homotopic** if they are $\underbrace{\text{homotopic rel } \{0,1\}}$

γ_0, γ_1 have same end-points at first

i.e., $\gamma_0(0) = \gamma_1(0)$; $\gamma_0(1) = \gamma_1(1)$.

If, in addition, both γ_0, γ_1 are loops, then they are **loop homotopic**. rel $\{0,1\}$

$\gamma_t(0)$

Must start with
$$\begin{array}{ccccc} \gamma_0(0) & = & H(0,t) & = & \gamma_1(0) \\ \| & & & & \| \\ \gamma_0(1) & = & H(1,t) & = & \gamma_1(1) \end{array}$$

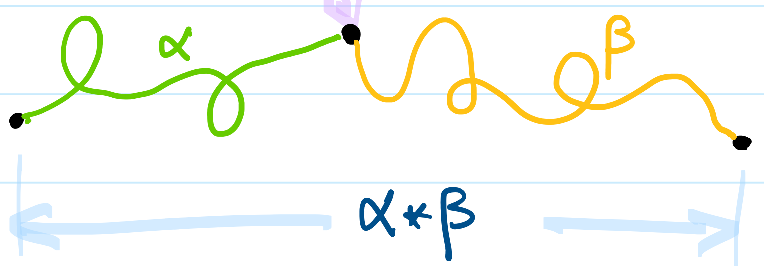
$\gamma_t(1)$

Theorem. Every theorem about homotopy has analogous versions for path / loop homotopy.

Here is a useful fact particularly for path or loop homotopy

Proposition. If $\varphi : [0,1] \longrightarrow [0,1]$ is a change of parameter, i.e., homeomorphism with $\varphi(0)=0$, $\varphi(1)=1$, then for any path $\gamma : [0,1] \longrightarrow X$, $\gamma \circ \varphi \simeq \gamma$ rel $\{0,1\}$.

Definition. Let $\alpha, \beta : [0,1] \longrightarrow X$ be paths in X with $\boxed{\alpha(1) = \beta(0)}$. Their concatenation $\alpha * \beta : [0,1] \longrightarrow X$ is a path defined by
$$\alpha*\beta(s) = \begin{cases} \alpha(2s), & s \in [0, 1/2] \\ \beta(2s-1), & s \in [1/2, 1] \end{cases}$$

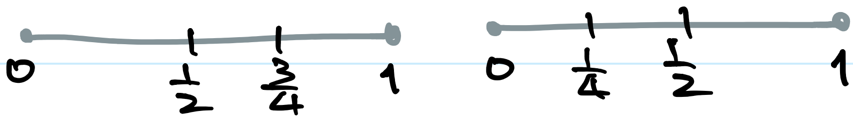


Geometrically, $\alpha * \beta$ is simply first travel α then travel β.

What about this
$$\sigma(s) = \begin{cases} \alpha(3s), & s \in [0, 1/3] \\ \beta(\frac{3}{2}s - \frac{1}{2}), & s \in [1/3, 1] \end{cases}$$

As mappings, $\sigma \neq \alpha * \beta$

Moreover, $\alpha * (\beta * \gamma) \neq (\alpha * \beta) * \gamma$



By above proposition, $\sigma \simeq \alpha * \beta$ rel $\{0, 1\}$

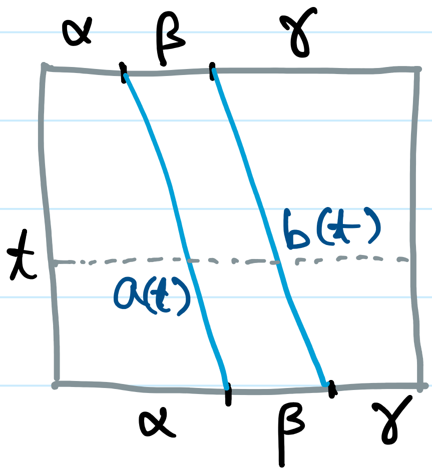
In fact, we have more useful results.

Proposition. Under well-defined conditions, if $\alpha_0 \simeq \alpha_1$ rel $\{0,1\}$ and $\beta_0 \simeq \beta_1$ rel $\{0,1\}$ then $\alpha_0 * \beta_0 \simeq \alpha_1 * \beta_1$ rel $\{0,1\}$.

Proof. Easy. Exercise.

Proposition. $\alpha * (\beta * \gamma) \simeq (\alpha * \beta) * \gamma$ rel $\{0,1\}$

Proof. Observe the schematic diagram

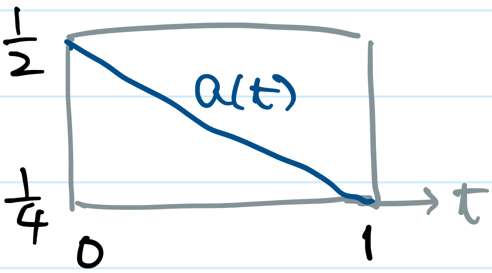


$t \in [0,1]$ for homotopy

$s \in [0,1]$, parameter for path

The homotopy $H : \underset{s}{[0,1]} \times \underset{t}{[0,1]} \longrightarrow X$

is
$$H(s,t) = \begin{cases} \alpha(\text{Exercise}), & s \in [0, a(t)] \\ \beta(\text{Exercise}), & s \in [a(t), b(t)] \\ \gamma(\text{Exercise}), & s \in [b(t), 1] \end{cases}$$

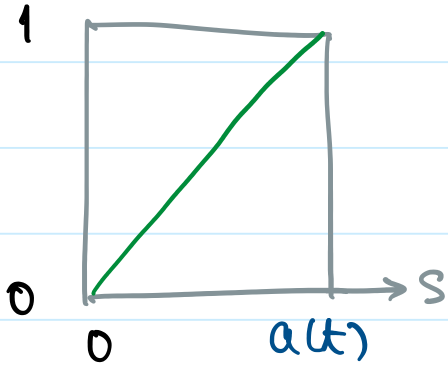


$$\therefore \quad \frac{a(t) - \frac{1}{2}}{t - 0} = \frac{-1}{4}$$

$$a(t) = \frac{-t}{4} + \frac{1}{2}$$

Similarly, $b(t)$ can be found.

Answer to Exercise for α:



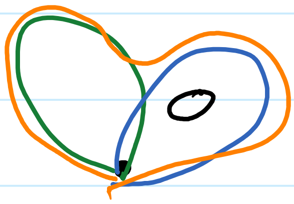
Thus, it should be

$$\alpha\left(\frac{s}{a(t)}\right).$$

Now, consider all loops $\gamma : [0,1] \longrightarrow X$ based at $x_0 \in X$, i.e., $\gamma(0) = x_0 = \gamma(1)$ under loop homotopy.

Definition. The fundamental group of X at x_0 is $\pi_1(X, x_0) = \left\{ \begin{array}{l} \text{loops in } X \\ \text{based at } x_0 \end{array} \right\} \Big/ \simeq \text{rel } \{0,1\}$

with $[\alpha] \cdot [\beta] \stackrel{\text{def}}{=} [\alpha * \beta]$



$\pi_1(X, x_0)$ has an associative (but may not be commutative) "multiplication".

We have established that

$$\pi_1(X,x_0) = \left\{ \begin{array}{l} \text{loops in } X \\ \text{based at } x_0 \in X \end{array} \right\} \Big/ \simeq \text{rel } \{0,1\}$$

has a "multiplication".

The following facts guarantee that $\pi_1(X,x_0)$ is truly a group.

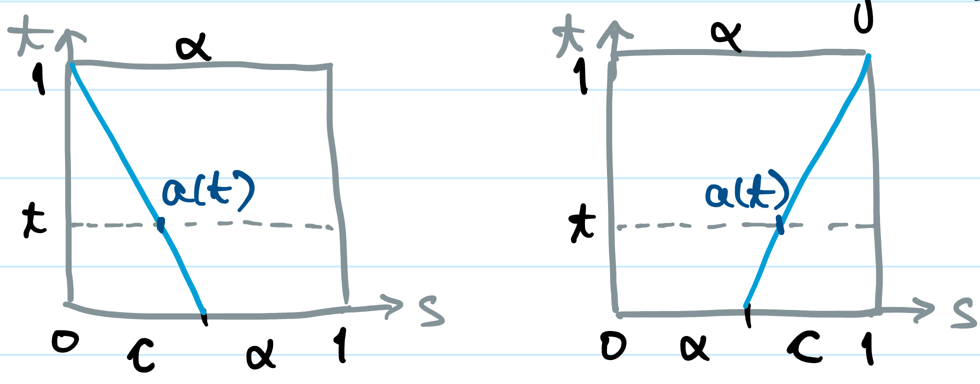
Proposition. Let $c : [0,1] \longrightarrow \{x_0\} \subset X$ be the constant loop. Then for each loop $\alpha : [0,1] \longrightarrow X$ based at x_0,

$$c * \alpha \simeq \alpha * c \simeq \alpha \quad \text{rel } \{0,1\}$$

That is, $[c]$ is the identity of $\pi_1(X,x_0)$

Main idea of proof.

Observe the two schematic diagrams



Proposition. For each loop $\alpha : [0,1] \longrightarrow X$ based at $x_0 \in X$, define $\bar{\alpha} : [0,1] \longrightarrow X$ by

$$\bar{\alpha}(s) = \alpha(1-s), \quad s \in [0,1].$$

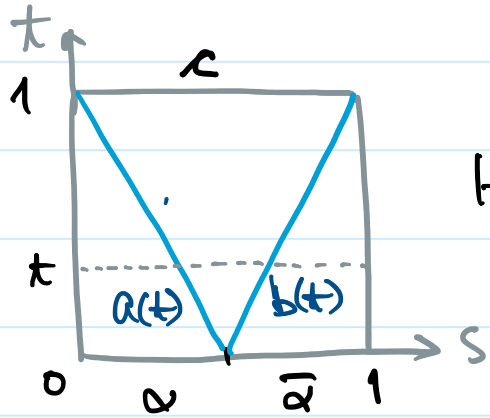
Then $\bar{\alpha}$ is a loop based at x_0 and

$$\alpha * \bar{\alpha} \simeq c \simeq \bar{\alpha} * \alpha \quad \text{rel } \{0,1\}$$

Equivalently, $[\bar{\alpha}] = [\alpha]^{-1}$ in $\pi_1(X, x_0)$

Proof. The rationale is a bit different.

First, observe the schematic diagram



$$H(s,t) = \begin{cases} \alpha(\quad), & s \in [0, a(t)] \\ x_0, & s \in [a(t), b(t)] \\ \bar{\alpha}(\quad), & s \in [b(t), 1] \end{cases}$$

If they run through $[0,1]$ as before, what is the geometric meaning?

fast α | at x_0 | fast $\bar{\alpha}$

stay long at x_0

almost infinite speed NOT continuous

Should travel as: part of α | x_0 | part of $\bar{\alpha}$

$a(t)$ $\qquad$ $b(t) = 1 - a(t)$

i.e., $\alpha(s)$ $\qquad x_0 \qquad$ $\bar{\alpha}(s)$