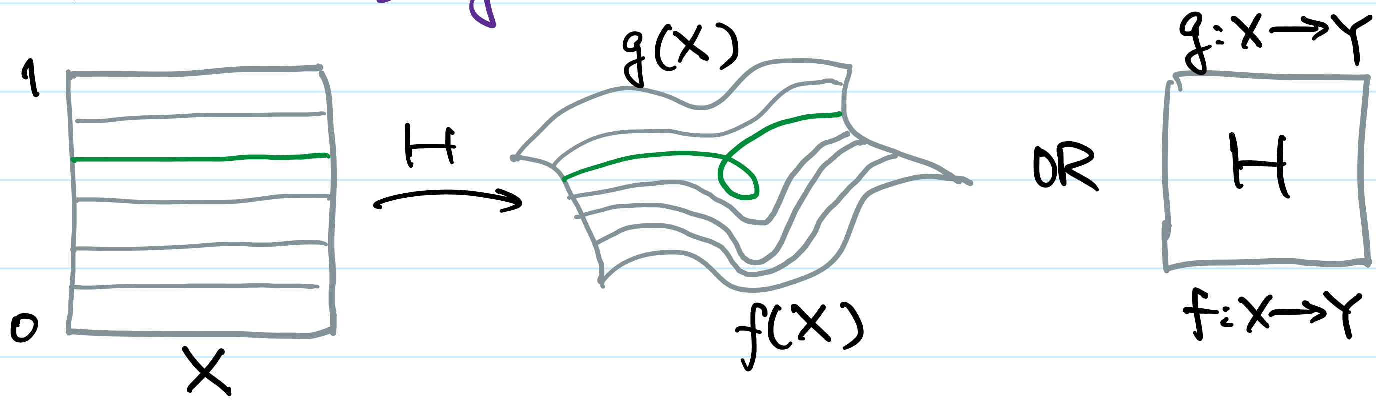
Homotopy Let X, Y be topological spaces. Two continuous mappings $f, g: X \longrightarrow Y$ are **homotopic** to each other if $\exists$ continuous $H: X \times [0,1] \longrightarrow Y$ such that $\forall\ x \in X$ $H(x,0) = f(x)$, $H(x,1) = g(x)$.
The mapping H is called a **homotopy** between f and g (or from f to g).

Notation: $f \simeq g$ or $f \overset{H}{\simeq} g$

Illustrative Diagram



Often, we denote $H_t : X \longrightarrow Y$, $t \in [0,1]$ where $H_t(x) = H(x,t)$, $x \in X$

Then $(H_t)_{t \in [0,1]}$ is a family of continuous mappings, continuously depending on t, which changes from f to g.

Example. Any two linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ are homotopic.

$$a_{ij} \rightsquigarrow a_{ij}(1-t) + b_{ij}\, t \rightsquigarrow b_{ij}$$

Example. If Y is path connected then any two constant mappings $X \longrightarrow Y$ are homotopic.

$$c_1 : X \longrightarrow \{y_1\} \subset Y$$
$$c_2 : X \longrightarrow \{y_2\} \subset Y$$
$$c_t : X \longrightarrow \{\gamma(t)\} \subset Y$$
$$\text{with } \gamma(0) = y_1,\ \gamma(1) = y_2$$

Definition. A mapping $X \longrightarrow Y$ is called null homotopic or homotopically trivial if it is homotopic to a constant mapping.

Example: Any $f : X \longrightarrow \mathbb{R}^n$ is null homotopic.

$$H(x,t) = t f(x), \quad x \in X, \quad t \in [0,1]$$
$$f \overset{H}{\simeq} c_0 : X \longrightarrow \{0\} \subset \mathbb{R}^n$$

Consequence. $id : \mathbb{R}^n \to \mathbb{R}^n$ is null homotopic

Exercise. If $id : Z \to Z$ is null homotopic then every $f : X \to Z$ is null homotopic. The converse is also true.

Remark. id is null homotopic indicates that the space is very "simple".

Proposition. Homotopy is an equivalence relation

"Reflexive" That is $f \overset{H}{\simeq} f$. Trivial.

$$H(x,t) = f(x) \quad \text{indep. of } t \in [0,1]$$

"Symmetric" That is $f \overset{H}{\simeq} g \Rightarrow g \overset{??}{\simeq} f$

We already have continuous

$$H : X \times [0,1] \longrightarrow Y$$
$$H(x,0) = f(x)$$
$$H(x,1) = g(x)$$
$$H' : X \times [0,1] \longrightarrow Y$$
$$H'(x,t) = H(x, 1-t)$$

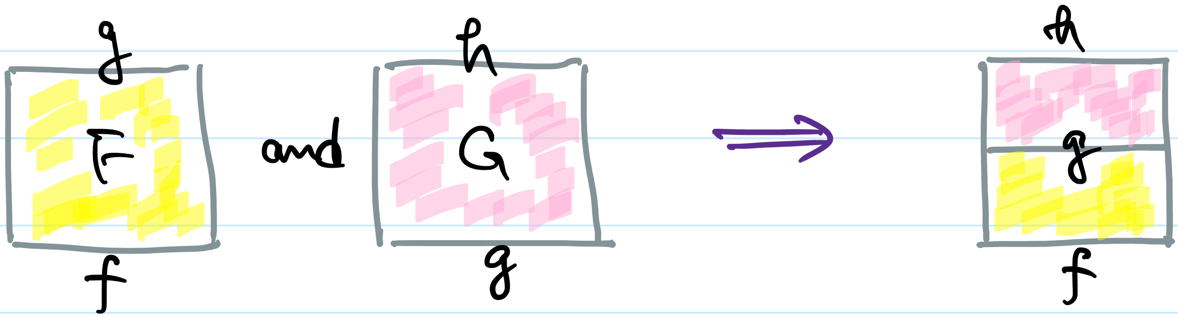
reversing $0 \nearrow 1$ to $1 \searrow 0$

"Transitive" That is $f \overset{F}{\simeq} g$ and $g \overset{G}{\simeq} h$

$$\Downarrow$$

$$f \overset{H?}{\simeq} h$$

Schematically, it is

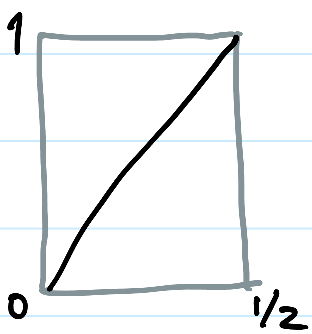


The key method is:

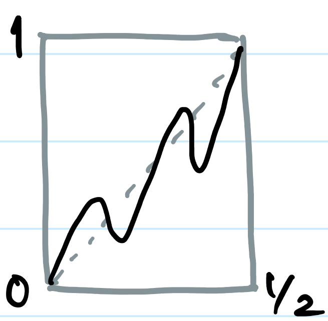
$[0,1] \xleftarrow{\text{re-scale}} [0,\tfrac{1}{2}]$

$[0,1] \xleftarrow{\text{and shift}} [\tfrac{1}{2},1]$

$$\therefore\ H(x,t) = \begin{cases} F(x, 2t) & t \in [0,\tfrac{1}{2}] \\ G(x, 2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$



but it could be



Homotopy is never unique.

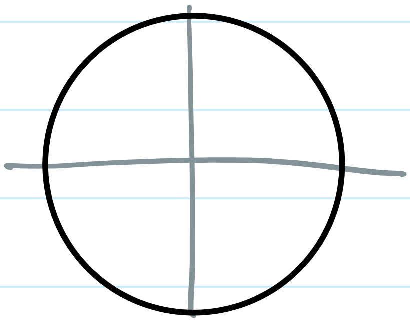
Example. $f, g, h : S^1 \longrightarrow \mathbb{R}^3$

$$e^{i\theta} \longmapsto (\cos\theta, \sin\theta, 0)$$

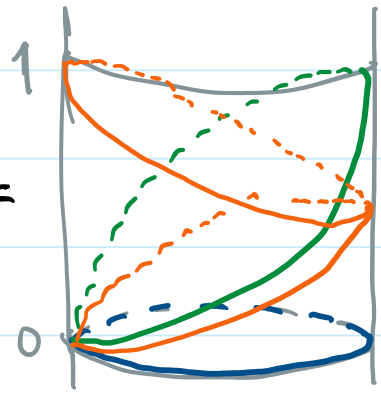
$$\theta \in [0, 2\pi] \qquad (\cos\theta, \sin\theta, \sin\tfrac{\theta}{2})$$

$$(\cos 2\theta, \sin 2\theta, \sin\tfrac{\theta}{2})$$

All three mappings are homotopic, in fact, null homotopic (due to $\mathbb{R}^3$)



$\longrightarrow S^1 \times \mathbb{R} =$



Exercise. $f \simeq g \not\simeq h$

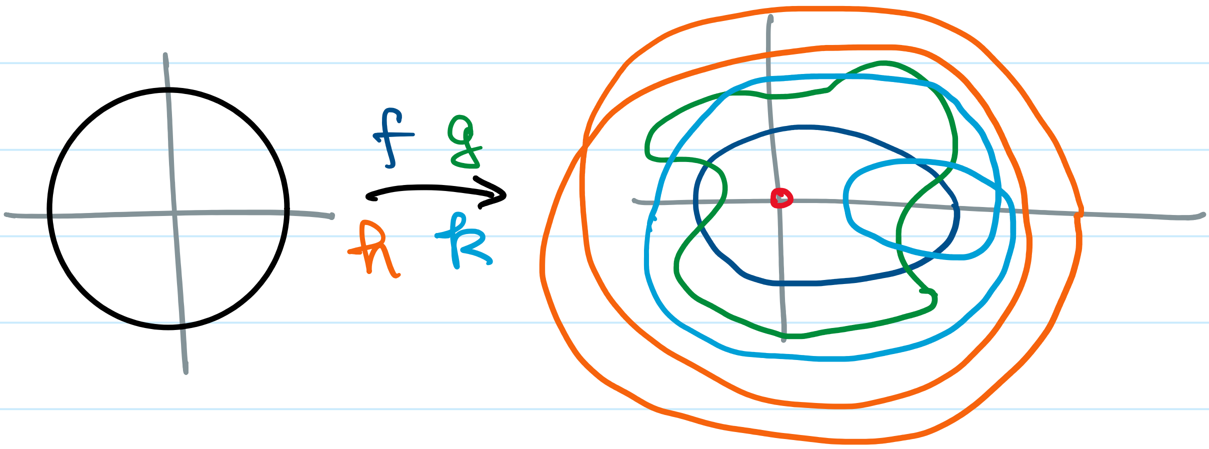
intuitive

$\underbrace{\hphantom{xxxxxxxxxxxx}}$

Somehow, this indicates

$\mathbb{R}^3 \not\cong S^1 \times \mathbb{R}$

Example from $S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$



It is expected that $\underbrace{f \simeq g \simeq k \not\simeq h}$

$\boxed{\text{This situation reflects that } \mathbb{R}^2 \setminus \{(0,0)\} \not\simeq \mathbb{R}^2}$

Homotopy Class Let X, Y be topological spaces; and $\mathcal{C}(X,Y)$ be the set of all continuous mappings from X to Y.
From above, homotopy, $\simeq$, is an equivalence relation on $\mathcal{C}(X,Y)$.

Denote $[X,Y] = \mathcal{C}(X,Y)/\simeq$ and for each continuous $f: X \longrightarrow Y$
$[f] \in [X,Y]$ is called the homotopy class of f.

Example. For all $n \geq 1$, for all space X,

$[X, \mathbb{R}^n]$ is a singleton

$$\|$$

$\{[c_0]\}$ where $c_0 : X \longrightarrow \{0\} \subset \mathbb{R}^n$

Example. In particular, take $X = S^1$

$[S^1, \mathbb{R}^2]$, $[S^1, \mathbb{R}^3]$ are singletons

$[S^1, \mathbb{R}^2 \setminus \{0\}]$, $[S^1, S^1 \times \mathbb{R}]$

seem not to be singletons

有乜用？

Obviously, we want to conclude

$$S^1 \times \mathbb{R} \not\cong \mathbb{R}^n, \quad n \geq 1$$

and $\mathbb{R}^2 \setminus \{0\} \not\cong \mathbb{R}^n, \quad n \geq 1$

Can we? What do we need logically?

Proposition.

- Let X, Y_1, Y_2 be topological spaces.
  If $Y_1 \simeq Y_2$ then $[X, Y_1] \xleftrightarrow[\text{bijective}]{} [X, Y_2]$
- Let X_1, X_2, Y be topological spaces.
  If $X_1 \simeq X_2$ then $[X_1, Y] \xleftrightarrow[\text{bijective}]{} [X_2, Y]$

How?

Naturally, for

$$[f_1] \in [X_1, Y] \text{ or } [X, Y_1]$$

$\downarrow$ find a suitable

$$[f_2] \in [X_2, Y] \text{ or } [X, Y_2]$$

Theorem (一石二鳥, in fact, 多鳥)

Let X, Y, Z be topological spaces.

If $f_0 \simeq f_1 : X \longrightarrow Y$, $g_0 \simeq g_1 : Y \longrightarrow Z$

then $g_0 \circ f_0 \simeq g_1 \circ f_1 : X \longrightarrow Z$

Example. $Y_1 \cong Y_2 \Rightarrow [X, Y_1] \xleftrightarrow[\text{bijective}]{} [X, Y_2]$

Homeomorphism
$h: Y_1 \to Y_2$

Take any $[f]$, i.e.,
$f: X \to Y_1$

Then $h \circ f : X \to Y_2$, $[h \circ f]$

Naturally, define

$$[X, Y_1] \xrightarrow{\varphi} [X, Y_2]$$
$$[f] \mapsto \varphi([f]) = [h \circ f]$$

What do we need to check?

* Well-defined: $[f_0] = [f_1] \Longrightarrow [h \circ f_0] = [h \circ f_1]$

$f_0 \simeq f_1$ $\qquad$ $h \circ f_0 \simeq h \circ f_1$

Use $g_0 = g_1 = h$
in Theorem

* Bijective: create an inverse

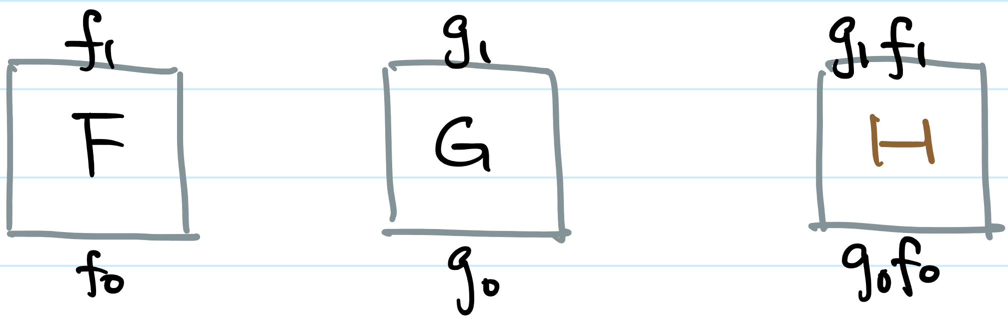
$$[h^{-1}\circ k] \in [X, Y_1] \xleftarrow{\psi} [k] \in [X, Y_2]$$

- Well-defined of ψ, need Theorem
- $\psi = \varphi^{-1}$, i.e., $\psi\varphi = id$ & $\varphi\psi = id$

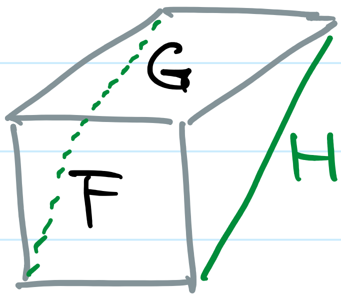
$$[f] \xrightarrow{\varphi} [h\circ f] \xrightarrow{\psi} [h^{-1} h f] = [f]$$

Proof of Theorem.

Wish: $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1 \Longrightarrow g_0 f_0 \overset{H?}{\simeq} g_1 f_1$



Idea comes from



$$H(x,t) = G(F(x,t), t)$$

Recall that $\mathbb{R}^n$ is exceptionally simple,

$[X, \mathbb{R}^n]$ is singleton $\forall$ space X

$\Updownarrow$

every mapping $f: X \to \mathbb{R}^n$ is null homotopic

$\Updownarrow$

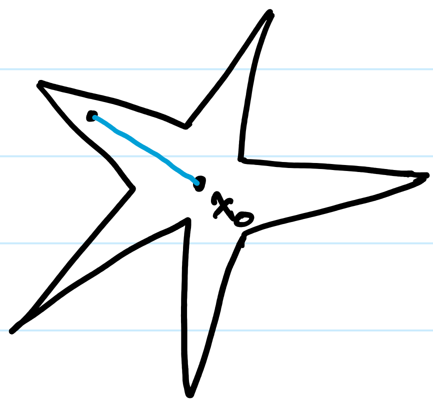
$id: \mathbb{R}^n \to \mathbb{R}^n$ is null homotopic

Definition. A space X is **contractible** if $id_X : X \longrightarrow X$ is null homotopic.

Examples of contractible spaces.

* A point $\{x_0\}$
* $\mathbb{R}^n$, $n \geq 1$
* A convex subset of $\mathbb{R}^n$
* A star-shape subset X of $\mathbb{R}^n$

$\exists\ x_0 \in X$ such that $\forall\ x \in X$ the straight line segment $\overline{x_0 x} \subset X$.



Definition. Two spaces X, Y are of the same homotopy type or homotopy equivalent ($X \simeq Y$) if $\exists$ continuous mappings $f: X \to Y$ and $g: Y \to X$ such that

$$g \circ f \simeq id_X : X \to X, \quad f \circ g \simeq id_Y : Y \to Y$$

In such case, f, g are called homotopy equivalences between X, Y and they are homotopy inverses to each other.

Example.

* If X, Y are homeomorphic then $X \simeq Y$

  Obvious, $g = f^{-1}$, so $g \circ f \equiv id_X$, $f \circ g \equiv id_Y$

* Any contractible space has the homotopy type of a point.

  $\therefore X \simeq Y \not\Rightarrow X = Y$

Proposition.

* If $X_1 \simeq X_2$ then $[X_1, Y] \longleftrightarrow [X_2, Y]$
* If $Y_1 \simeq Y_2$ then $[X, Y_1] \xleftrightarrow{\text{bijective}} [X, Y_2]$

Proof. A consequence of 一石多鳥