# Lect19-20180321

Tuesday, March 20, 2018 2:24 PM

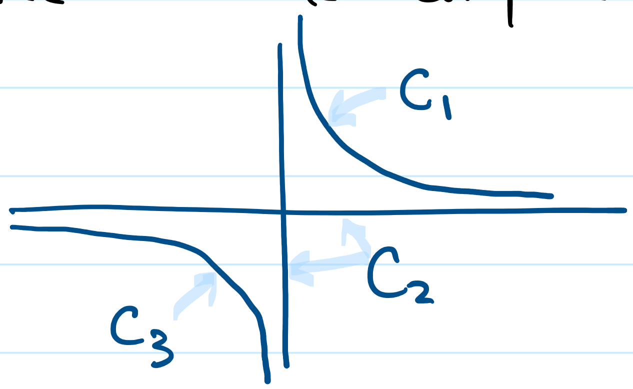
**Definition.** Let $x_0 \in X$. $C \subset X$ is the connected component of $x_0$ if either one holds.

① $C$ is the largest connected subset of $X$ containing $x_0$.

② $C = \bigcup \{A \subset X : x_0 \in A \text{ and } A \text{ is connected}\}$

③ Let $\sim$ be an equivalence relation on $X$ where $x \sim y$ if $x, y$ belong to a connected subset. $C = [x_0]$, the equivalence class of $x_0$.

**Example.**
$X = \{(x,y) \in \mathbb{R}^2 : xy = 0 \text{ or } xy = 1\} \subset \mathbb{R}^2$ has three connected components




What is needed about 3 definitions?
They are equivalent.

②⟺③ Trivial

$x \in C_2 \Leftrightarrow \exists$ connected $A$ with $x_0 \in A$, $x \in A$

$\Leftrightarrow x \sim x_0 \Rightarrow x \in [x_0]$

Lect19-p2

Mar 22, Thursday, 2018 3:28 PM

① $\Rightarrow$ ② Trivial

$C_1$ = largest connected subset containing $x_0$

$C_2 = \bigcup \underbrace{\{\text{connected } A \subset X : x_0 \in A\}}_{\mathcal{A}}$

By def. of $C_1$, $C_1 \in \mathcal{A}$, $\therefore$ $C_1 \subset \bigcup \mathcal{A}$

Every $A \in \mathcal{A}$ satisfies $A \subset C_1$, $\therefore$ $\bigcup \mathcal{A} \subset C_1$

$\therefore$ $C_1 = C_2$

② $\Rightarrow$ ① Easy

Only need to show $\bigcup \mathcal{A}$ is connected.

Theorem. Let $A_\alpha \subset X$ be connected subsets.

If $\forall$ pair $\alpha, \beta \in I$, $A_\alpha \cap A_\beta \neq \phi$

then $\bigcup\limits_{\alpha \in I} A_\alpha$ is connected.

Remark. In Definition ②,

$$\mathcal{A} = \{A \subset X : A \text{ is connected}, x_0 \in A\}$$

$$A_\alpha, A_\beta \in \mathcal{A} \Longrightarrow A_\alpha \cap A_\beta \supset \{x_0\} \neq \phi$$

Idea of proof. Let $S \subset \bigcup\limits_{\alpha \in I} A_\alpha$ be both open and closed.

Wish: $S = \phi$ or $S = \bigcup\limits_{\alpha \in I} A_\alpha$

Lect19-p3

Tuesday, 20 March 2018 4:11 PM

Obviously, by considering $S \cap A_\alpha \ \forall \alpha \in I$

we have $S \cap A_\alpha = \phi$ or $S \cap A_\alpha = A_\alpha$

$$\therefore \bigcup_{\alpha \in I} (S \cap A_\alpha) = \phi \quad \text{or} \quad \bigcup_{\alpha \in I} (S \cap A) = \bigcup_{\alpha \in I} A_\alpha$$

$$S \cap \left( \bigcup_{\alpha \in I} A_\alpha \right) = S$$

WRONG above

$$\forall \alpha \in I \left[ S \cap A_\alpha = \phi \quad \text{or} \quad S \cap A_\alpha = A_\alpha \right]$$

some $\alpha$ other $\alpha$

We actually need :

$$\left[ \forall \alpha \in I \ S \cap A_\alpha = \phi \right] \text{or} \left[ \forall \alpha \in I \ S \cap A_\alpha = A_\alpha \right]$$

Assume $\exists \ \alpha \in I$ with $S \cap A_\alpha = \phi$

Let $\beta \in I$, we already know

$$S \cap A_\beta = \phi \quad \text{or} \quad \underbrace{S \cap A_\beta = A_\beta}_{\text{Rule out}}$$

$$A_\alpha \cap (S \cap A_\beta) = A_\alpha \cap A_\beta \neq \phi$$

$$\phi \cap A_\beta = (A_\alpha \cap S) \cap A_\beta$$

Contradiction

**Theorem.** If $X, Y$ are connected then so is $X \times Y$.

**Idea of proof.**

For each $(a,b) \in X \times Y$,

$X \times \{b\}$, $\{a\} \times Y$ are connected

and $(X \times \{b\}) \cap (\{a\} \times Y) = \{(a,b)\} \neq \emptyset$

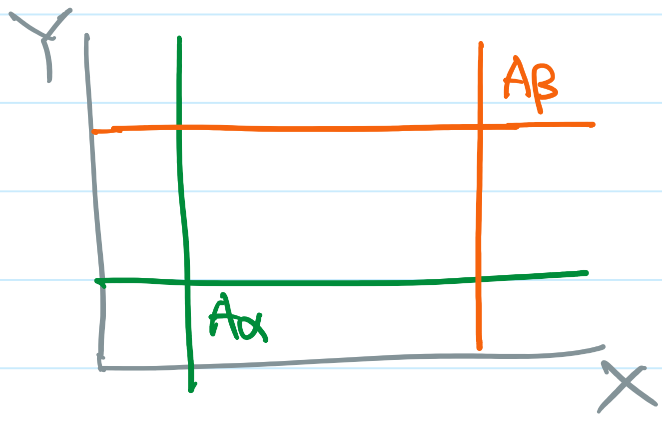
$\therefore$ $(X \times \{b\}) \cup (\{a\} \times Y)$ is connected

Let $A_\alpha = (X \times \{b\}) \cup (\{a\} \times Y)$, $\alpha = (a,b) \in X \times Y$

Then $A_\alpha \cap A_\beta \neq \emptyset$ $\forall$ pair $\alpha, \beta$

$X \times Y = \bigcup_\alpha A_\alpha$

is connected



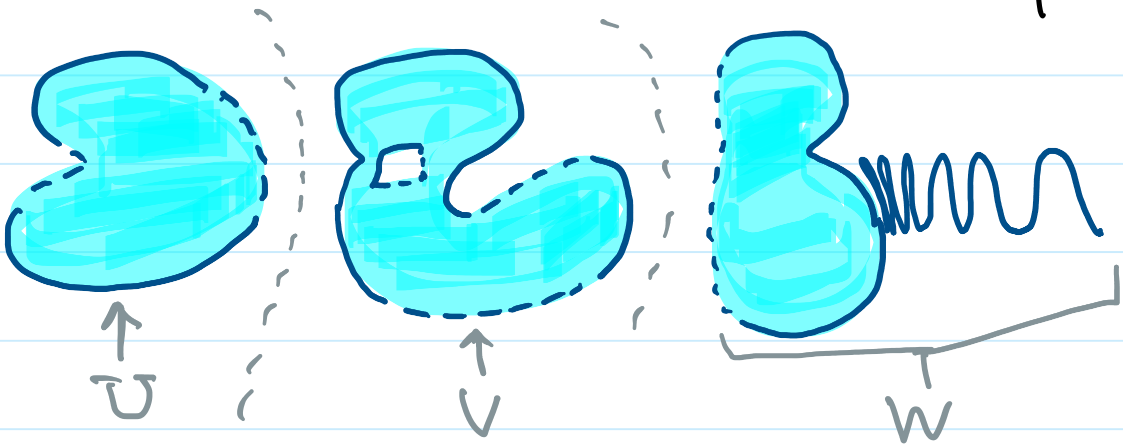

**Fact.** True for infinite product, but the proof is harder.

See a supplement later.

# Lect19-p5

Tuesday, 20 March 2018 5:12 PM

**Intuition** X has several connected components




$$X = U \cup (V \cup W)$$

both open & closed

Again, both open & closed

Apparently, every connected component of X is both open and closed in X.

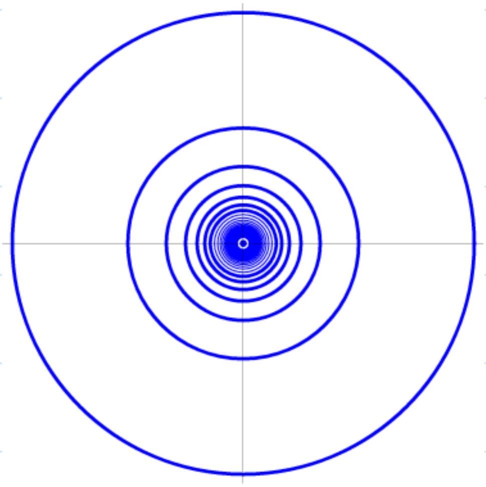
wrong true

**Example.** $X = \bigcup_{n=0}^{\infty} C_n \subset \mathbb{R}^2$ where

$$C_n = \{(x,y) : x^2 + y^2 = \tfrac{1}{n^2}\},$$

$1 \leq n \in \mathbb{N}$

$$C_0 = \{(0,0)\}$$



Each $C_n$, $n \geq 1$, is both open & closed while $C_0$ is only closed but not open

**Theorem** Let $A \subset X$ be a connected set. If $A \subset B \subset \bar{A}$ then $B$ is also connected.

**Proof.** Let $S$ be both open & closed in $B$

$$\exists\ G \in \mathcal{J}_X \quad \text{and} \quad X \setminus F \in \mathcal{J}_X$$

$$S = G \cap B \quad = \quad F \cap B$$

$$\therefore\ S \cap A = \underbrace{G \cap A}_{} \quad = \quad \underbrace{F \cap A}_{}$$

open & closed in $A$

By connectedness of $A$,

$$S \cap A = \phi \quad \text{or} \quad S \cap A = A$$

How to get from $A$ to $\bar{A}$?

$$G \cap A = \phi \quad \text{or} \quad F \cap A = A$$

$$\therefore\ A \subset \underbrace{X \setminus G}_{} \quad \text{or} \quad A \subset \underbrace{F}_{}$$

closed sets

$$\therefore\ \bar{A} \subset X \setminus G \quad \text{or} \quad \bar{A} \subset F$$

$$S \subset B \subset \bar{A}, \quad S = G \cap B = \phi \qquad \text{or} \qquad S \subset B \subset \bar{A}, \quad S = F \cap B = B$$

**Consequently,** every connected component must be closed, by maximality