Lect11-20180214

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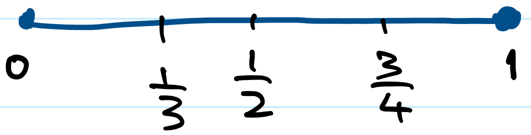
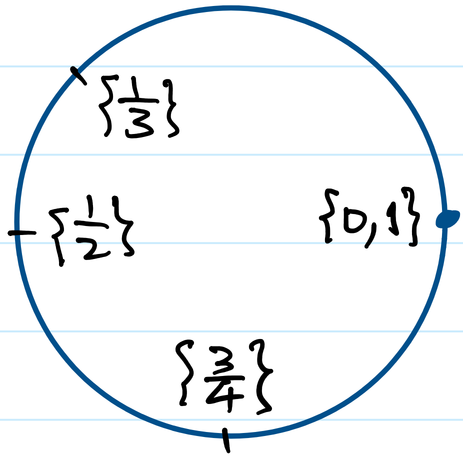
Glue to form a circle.

Take the closed interval $[0,1]$ with $\mathcal{J}_{std}$ $\Rightarrow$ Make a circle



Glue the end-pts



Mathematically, done by an equivalence relation.

For $s,t \in [0,1]$, $s \sim t$ if $|s-t| = 0$ or 1.

only cases are
$s=t$ or $s=0, t=1$ or $s=1, t=0$

What is the quotient set?

$$[0,1]/\sim \; = \big\{ \{0,1\} \big\} \cup \big\{ \{s\} : 0<s<1 \big\}$$

Two end-pts become one

others are still "single"

This is a partition of $[0,1]$, i.e.,

* the sets in it are disjoint
* their union is $[0,1]$

Remark. From set theory, the quotient set is a partition of X. Equivalently, it determines the relation $\sim$.

What is the quotient map ?

$$[0,1] \xrightarrow{q} [0,1]/\sim$$

$$s \in [0,1] \longmapsto [s], \text{ its equivalence class}$$

$$0 \longmapsto \{0,1\}$$

$$0<s<1 \longmapsto \{s\}$$

$$1 \longmapsto \{0,1\}$$

Remark ① q is always surjective

② Any surjective map $X \longrightarrow$ any set defines an equivalence relation on X.

Exercise. (about set theory)

Given a set X and an equivalence relation $\sim$ on X. Consider the quotient map $q: X \longrightarrow X/\sim$.

For $y_1, y_2 \in X/\sim$, what is

$$q^{-1}(\{y_1, y_2\}) ?$$

Lect11-p3

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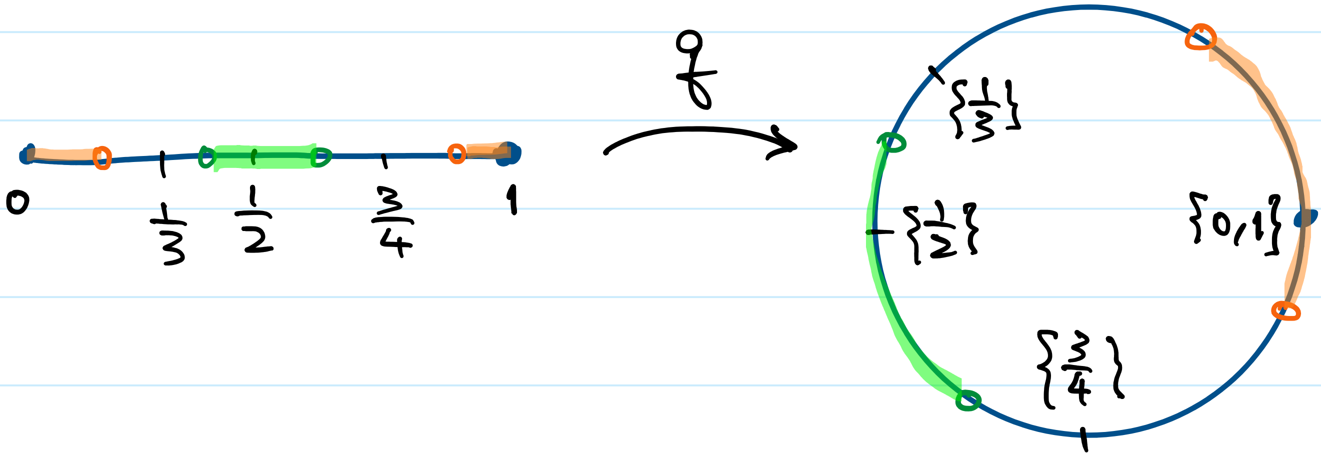
Topology of $[0,1]/\sim$. The drawing of $[0,1]/\sim$ as a circle already has hidden information.



The topology of $[0,1]$ is defining open sets on $[0,1]/\sim$, i.e., a topology for $[0,1]/\sim$

Definition. Given $(X, \mathcal{T}_X)$ and either an equivalence relation $\sim$ on X or a surjective mapping $q: X \longrightarrow Q$.
The quotient topology for $X/\sim$ or Q is $\mathcal{T}_q = \{V \subset X/\sim \text{ or } Q : q^{-1}(V) \in \mathcal{T}_X\}$

Warning.
$\mathcal{T}_q \neq \{q(U) \subset X/\sim \text{ or } Q : U \in \mathcal{T}_X\}$

Lect11-p4

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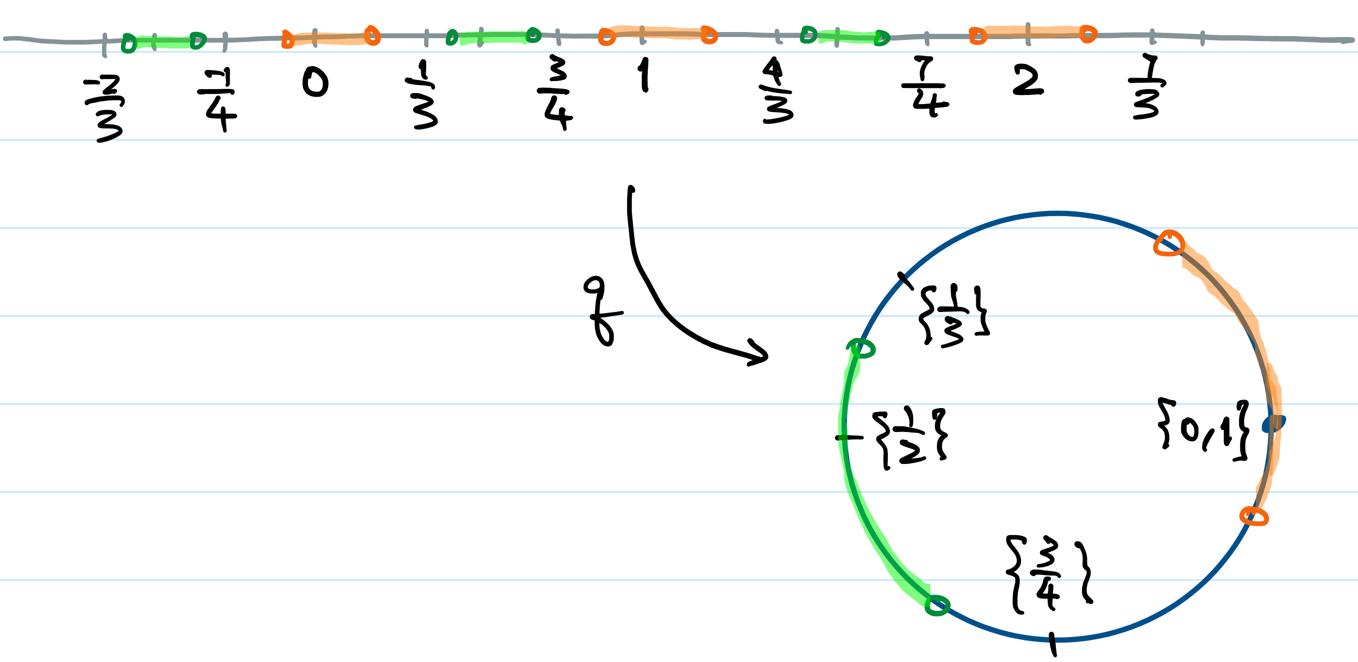
Example of Circle.

1. Seen as $[0,1]/\sim$ as above.
2. Define $\sim$ on $\mathbb{R}$, $\underbrace{x \sim y \text{ if } x-y \in \mathbb{Z}}$

In group theory, $\mathbb{R}/\sim$ is the factor group, $\mathbb{R}/\mathbb{Z}$



3. Both homeomorphic to $S^1 \subset (\mathbb{R}^2, \mathcal{T}_{std})$

$$[0,1]/\sim \;\longleftrightarrow\; \mathbb{R}/\mathbb{Z} \;\longleftrightarrow\; S^1$$

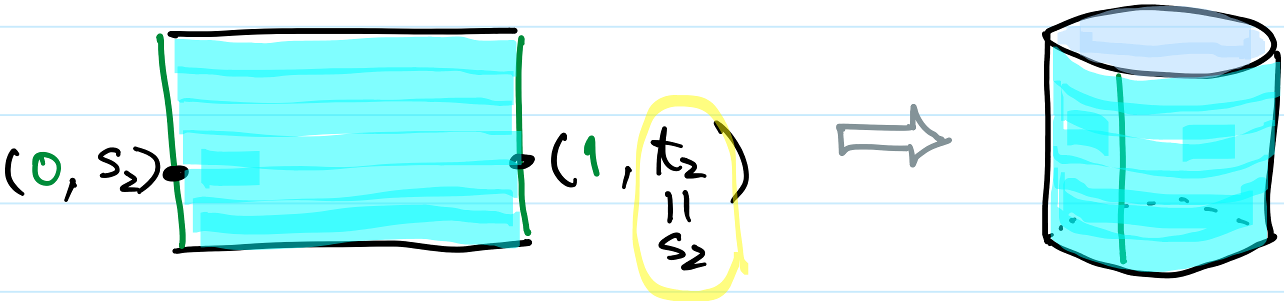
$$[s] \;\longleftrightarrow\; s+\mathbb{Z} \;\longleftrightarrow\; e^{2\pi i s}$$

Cylinder = Annulus

- $\left([0,1]\times[0,1]\right)/\sim$ where

  $s=(s_1,s_2)\sim t=(t_1,t_2)$ if $s=t$ or

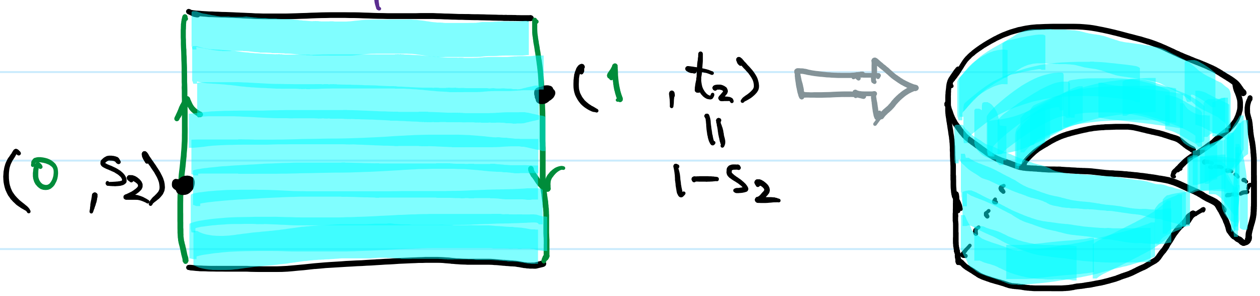
  $|s_1-t_1|=1$ and $s_2=t_2$



Equivalently,

- $S^1\times[0,1]=\left([0,1]/\sim\right)\times[0,1]=\mathbb{R}/\mathbb{Z}\times[0,1]$

Möbius Strip



$\left([0,1]\times[0,1]\right)/\sim$ where

$s=(s_1,s_2)\sim t=(t_1,t_2)$ if $s=t$ or

$|s_1-t_1|=1$ and $t_2=1-s_2$

Glue horizontal edges

Flip vertical edges

Lect11-p6

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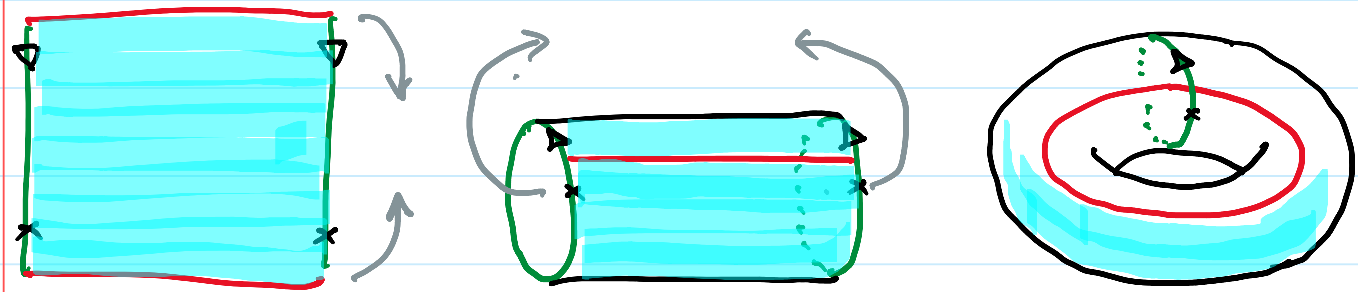
Torus



How to define the equivalence relation?

Apparently, a two-step process on

$$\left(\left([0,1]\times[0,1]\right)/\sim\right)/\approx$$

First: $(s_1,s_2)\sim(t_1,t_2)$ by $\begin{cases}|s_1-t_1|=0,1\\ s_2=t_2\end{cases}$

Second: $[(s_1,s_2)]\approx[(t_1,t_2)]$ by $\begin{cases}s_1=t_1\\ |s_2-t_2|=0,1\end{cases}$

Question. What about doing in one-step.

$(s_1,s_2)\sim(t_1,t_2)$ by $\begin{cases}|s_1-t_1|=0,1\\ |s_2-t_2|=0,1\end{cases}$?

Is the following true?

Let X be a set and $\sim$ be an equivalence relation on X and $\approx$ be an equivalence relation on $X/\sim$. Then $\exists$ an equivalence relation $\simeq$ on X such that

$$(X/\sim)/\approx \xleftrightarrow[\text{bijection}]{} X/\simeq$$

Lect11-p7

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Language

- $[0,1]/\sim \; = S^1$ Identify 0, 1 in $[0,1]$
- Cylinder Identify $(0,t)$ and $(1,t)$ in $[0,1]\times[0,1]$
- Möbius strip Identify $(0,t)$ and $(1,1-t)$ in $[0,1]\times[0,1]$
- Torus Identify $(0,t)$ with $(1,t)$ and $(s,0)$ with $(s,1)$ in $[0,1]\times[0,1]$