Lect10-20180212

Monday, 5 February 2018 10:43 AM

Finite Product $X \times Y$

$\mathcal{T}_{X\times Y}$ is generated by

$$S = \{U \times Y : U \in \mathcal{T}_X\} \cup \{X \times V : V \in \mathcal{T}_Y\}$$

That is having a base

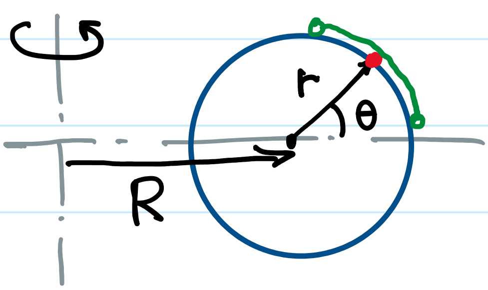
$$\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

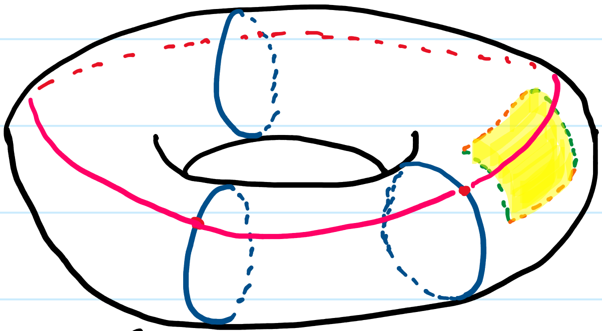
Examples

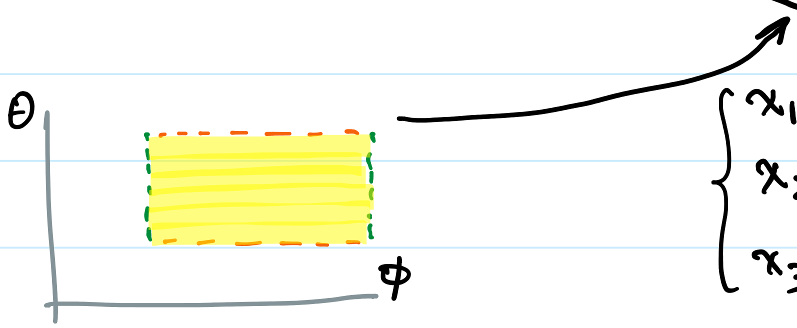
* $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$
* Annulus $=$ Cylinder $=$ $S^1 \times [a,b]$ product
  - Annulus $\cap$ $\mathbb{R}^2$; Cylinder $\cap$ $\mathbb{R}^3$; product $\cap$ $\mathbb{R}^2 \times \mathbb{R}$
* S^2 not a product
* Möbius strip not a product

as surfaces. Proof nontrivial

Torus, $\mathbb{T}$ = surface of revolution $\subset \mathbb{R}^3$







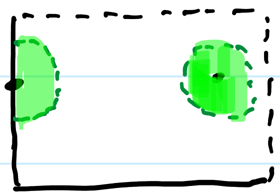
$$\begin{cases} x_1 = (R + r\cos\theta)\cos\phi \\ x_2 = (R + r\cos\theta)\sin\phi \\ x_3 = r\sin\theta \end{cases}$$

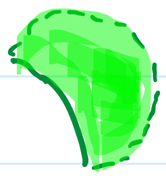
Lect10-p2

Feb 13, Tuesday, 2018 9:52 AM

If $\begin{cases} \theta \in (0, 2\pi) \\ \phi \in (0, 2\pi) \end{cases}$ then 1-1 but not onto

If $\begin{cases} \theta \in [0, 2\pi) \\ \phi \in [0, 2\pi) \end{cases}$ then not homeomorphic





No such open set in T^2

Homeomorphism to Torus

$$(e^{i\theta}, e^{i\phi}) \longrightarrow T^2$$

$$\cap$$

$S^1 \times S^1$ = A product of $S^1 \subset \mathbb{R}^2$

n-Torus

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}$$

Infinite Product Set

Given sets X_α, $\alpha \in I$, we have $x \in \prod_{\alpha \in I} X_\alpha$

where $x : I \longrightarrow \bigcup_{\alpha \in I} X_\alpha$ such that $x(\alpha) \in X_\alpha$

Examples

* $X_1 = A$, $X_2 \in B$, $x \in A \times B$ satisfies

$$x : \{1, 2\} \longrightarrow A \cup B \qquad \begin{cases} x(1) \in A \\ x(2) \in B \end{cases}$$

* $X_1 = X_2 = \cdots = X_n = \mathbb{R}$, $x \in \mathbb{R}^n$ if

$$x : \{1, 2, \cdots, n\} \longrightarrow \mathbb{R} = \mathbb{R} \cup \mathbb{R} \cup \cdots \cup \mathbb{R}$$

$$x(1), x(2), \cdots, x(n) \in \mathbb{R}$$

denote

$$x = (x_1, x_2, \cdots, x_n)$$

* If all $X_\alpha = Y$ then $x \in \prod_{\alpha \in I} Y$ means

$$x : I \longrightarrow Y$$

Thus $\prod_{\alpha \in I} Y = Y^I$

* $I = \mathbb{N}$, $X_\alpha = \{0,1\}$, $\prod_{\alpha \in \mathbb{N}} \{0,1\} = \{0,1\}^{\mathbb{N}}$

$x \in \{0,1\}^{\mathbb{N}}$ is an infinite sequence with entries 0, 1

For a finite product $X_1 \times X_2 \times \cdots \times X_n$, the generating set is

$$\bigcup_{k=1}^{n} \{ \underbrace{X_1 \times \cdots \times X_{k-1} \times U_k \times X_{k+1} \times \cdots \times X_n}_{} : U_k \in \mathcal{J}_k \}$$

How to rewrite it to a simple version?

$$\| $$

$$\pi_k^{-1}(U_k) \quad \text{where}$$

$$\pi_k : X_1 \times \cdots\cdots \times X_n \longrightarrow X_k \qquad \text{projection}$$

$$(x_1, \cdots\cdots, x_n) \longmapsto x_k$$

Definition. Given $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in I$, the **product topology** $\mathcal{T}_\Pi$ for $\prod_{\alpha \in I} X_\alpha$ is generated by

$$\mathcal{S} = \bigcup_{\alpha \in I} \{ \pi_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{T}_\alpha \}$$

After finite intersections, do we get

$$\left\{ \prod_{\alpha \in I} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \right\} ?$$

$$\begin{array}{ccc} \mathcal{B}_{Box} & \supsetneq & \mathcal{B}_\Pi \\ \downarrow & & \downarrow \\ \mathcal{T}_{Box} & \supsetneq & \mathcal{T}_\Pi \end{array}$$

All but finite factors are whole space X_α

Coincide when I is finite

Example. Let $I = \mathbb{N}$, $X_\alpha = (\{0,1\}, \text{discrete})$ and $\bar{0} = (0,0,0,\cdots,0,\cdots\cdots) \in \{0,1\}^{\mathbb{N}}$

In $\mathcal{T}_{Box}$, what are the nbhds of $\bar{0}$?

Quite a lot !!!

$\{0,1\}^{\mathbb{N}}$, $\{0\} \times \{0,1\}^{\mathbb{N}-1}$, $\{(0,0)\} \times \{0,1\}^{\mathbb{N}-2}$, ...

or $\{0\} \times \{0,1\} \times \{0\} \times \{0,1\}^{\mathbb{N}-3}$, ...

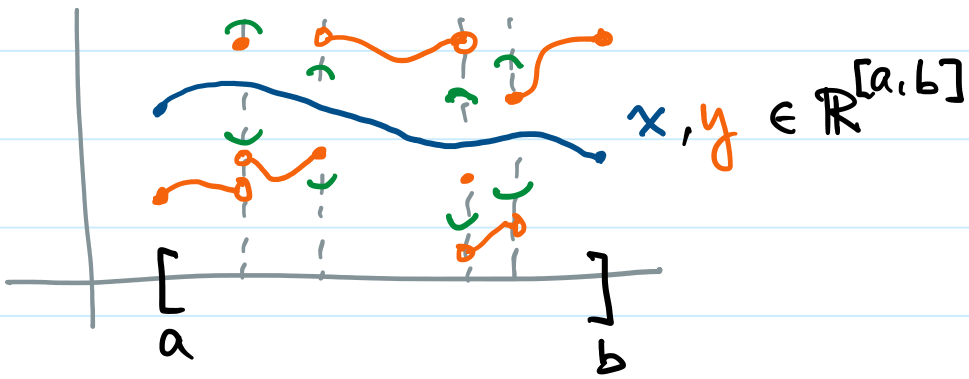
What is the smallest nbhd of $\bar{0}$?

Answer. $\{\bar{0}\}$

In $\mathcal{T}_\Pi$, is there a smallest nbhd of $\bar{0}$?

$$\{(0,0,\cdots,0)\}\times\{0,1\}^{\mathbb{N}-1000000}$$
$$\cup$$
$$\{(0,0,\cdots,0,0)\}\times\{0,1\}^{\mathbb{N}-1000001} \quad \text{or etc.}$$
$$\cup$$
$$\vdots$$

Example. $I=[a,b]$; $X_t=\mathbb{R}_{std}$ for all $t\in[a,b]$.



$y \in \varepsilon$-nbhd of $x \iff$
$|y(t)-x(t)|<\varepsilon$ for finitely many $t\in[a,b]$

Why use $\mathcal{T}_\Pi$ but not $\mathcal{T}_{BOX}$?

We have a lot of choices for $\prod_{\alpha\in I} X_\alpha = P$,
$$\{\phi,P\}\subset\cdots\cdots\subset\mathcal{T}_\Pi\subset\cdots\subset\mathcal{T}_{BOX}\subset\cdots\cdots\subset\mathcal{P}(P).$$

Most natural mappings $\pi_\alpha : P \longrightarrow X_\alpha$

Which $\mathcal{T}$ is good!

$$
\left.\begin{array}{ccc}
(P, \mathcal{P}(P)) & \longrightarrow & X_\alpha \\
\vdots & & \vdots \\
(P, \mathcal{T}_{Box}) & \longrightarrow & X_\alpha \\
\cup & & \vdots \\
(P, \mathcal{T}_\Pi) & \longrightarrow & X_\alpha
\end{array}\right] \text{continuous}
$$

$$
\left.\begin{array}{ccc}
\cup & & \\
\vdots & & \\
(P, \{\emptyset, P\}) & \longrightarrow & X_\alpha
\end{array}\right] \text{usually not continuous}
$$

Theorem. $\mathcal{T}_\Pi$ is the smallest topology for $\prod_{\alpha\in I} X_\alpha$ such that for each $\beta \in I$

$\pi_\beta : \prod_{\alpha\in I} X_\alpha \longrightarrow X_\beta$ is continuous

How to prove it ??

Answer. Simply by definition of $\mathcal{T}_\Pi$, which is generated by $\underbrace{\pi_\beta^{-1}(V_\beta),\ V_\beta \in \mathcal{T}_\beta}$

so belong to $\mathcal{T}_\Pi$

Lect10-p7

Sunday, 11 February 2018 12:14 PM

Theorem. Let $W \xrightarrow{f} P = \prod_{\alpha \in I} X_\alpha \xrightarrow{\pi_\beta} X_\beta$.

f is continuous $\Leftrightarrow$ $\forall \beta \in I$, $\underbrace{\pi_\beta \circ f}_{\text{coordinate function}}$ is so.

Useful: $(x,y) \overset{f}{\mapsto} (xy\sin(x+y), (x+y)e^{xy}, x^2-y^2)$

π_1: $xy\sin(x+y)$ π_2: $(x+y)e^{xy}$ π_3: x^2-y^2

"$\Rightarrow$" Trivial — composition of continuous mappings

"$\Leftarrow$" To verify continuity of

$$f : (W, \mathcal{J}_W) \longrightarrow (P, \mathcal{J}_\pi)$$

Where should we start?

Take any $G \in \mathcal{B}_\pi$.

Then, what do we wish?

$$f^{-1}(G) \in \mathcal{J}_W$$

$$\parallel$$

$$f^{-1}\left(\bigcap_{k=1}^{n} \pi_{\beta_k}^{-1}(U_{\beta_k})\right), \quad U_{\beta_k} \in \mathcal{J}_{\beta_k}$$

$$= \bigcap_{k=1}^{n} f^{-1}\left(\pi_{\beta_k}^{-1}(U_{\beta_k})\right)$$

$$= \bigcap_{k=1}^{n} \underbrace{(\pi_{\beta_k} \circ f)^{-1}(U_{\beta_k})}_{\text{by continuity of } \pi_\beta \circ f} \quad \square$$

Lect10-p8

Feb 13, Tuesday, 2018 10:06 AM

$$(W, \mathcal{J}_W)$$
$$\Big\downarrow f$$

Which ones make f continuous?

Above $\Rightarrow$ includes $\mathcal{J}_\Pi$

Also includes $\{\phi, P\}$, trivial

$$\overbrace{\{\phi, P\} \subset \cdots \subset (P, \mathcal{J}_\Pi)}^{?} \underbrace{\subset \cdots \subset \mathcal{J}_{Box} \subset \cdots \cdots \subset \mathcal{P}(P)}_{\text{make } \pi_\beta \text{ continuous}}$$

$$(P, \mathcal{J}_\Pi) \xrightarrow{\pi_\beta} X_\beta$$

Theorem. $\mathcal{J}_\Pi$ is the maximal topology on $P = \prod_{\alpha \in I} X_\alpha$ such that $f : (W, \mathcal{J}_W) \longrightarrow (P, \mathcal{J})$ is continuous

$\Leftrightarrow \forall \beta \in I \quad \pi_\beta \circ f : W \to X_\beta$ is so.

From above $\mathcal{J}_\Pi$ satisfies the property.

Maximality Let $\mathcal{J}$ on P have the above property.

Wish. $\mathcal{J} \subset \mathcal{J}_\Pi$

Simply consider $id : (P, \mathcal{J}_\Pi) \to (P, \mathcal{J})$

As $\pi_\beta \circ id = \pi_\beta$ is continuous, so is id.

Thus, $\forall G \in \mathcal{J}$, $(id)^{-1}(G) = G \in \mathcal{J}_\Pi$.

Example. Let $I = \mathbb{N}$; $(X_k, \mathcal{J}_k) = (\mathbb{R}, std)$, $k \in I$

Consider $f : (\mathbb{R}, std) \longrightarrow \mathbb{R}^{\mathbb{N}} = \prod_{k \in I} X_k$

$$t \longmapsto (t, t, \cdots, t, \cdots)$$

i.e. $f(t)_{(k)} = t \quad \forall k \in I$

For $f : (\mathbb{R}, \mathcal{T}_{std}) \longrightarrow (\mathbb{R}^N, \mathcal{T}_\pi)$

Is it continuous?

Yes, each $k \in I$, $\pi_k \circ f : (\mathbb{R}, std) \longrightarrow (\mathbb{R}, std)$

$t \mapsto t$

In fact, $f : (\mathbb{R}, std) \longrightarrow (\mathbb{R}^N, \uparrow\ )$ is continuous for $\{\phi, \mathbb{R}^N\} \subset \cdots \subset \cdots \subset \mathcal{T}_\pi$

What about $f : (\mathbb{R}, std) \longrightarrow (\mathbb{R}^N, \mathcal{T}_{BOX})$?

Not continuous! How to argue?

Choose $V \in \mathcal{T}_{BOX}$ such that $f^{-1}(V) \notin \mathcal{T}_{std}$.

$$V = (-1,1) \times \left(\tfrac{-1}{2}, \tfrac{1}{2}\right) \times \left(\tfrac{-1}{3}, \tfrac{1}{3}\right) \times \cdots \times \left(\tfrac{-1}{n}, \tfrac{1}{n}\right) \times \cdots\cdots$$

$$\rotatebox{90}{$\in$}$$

$$f(0) = (0, 0, 0, \cdots, 0, \cdots\cdots)$$

Suppose $f^{-1}(V)$ is open, Since $0 \in f^{-1}(V)$, then what?

$\exists\ \varepsilon > 0$, $0 \in (-\varepsilon, \varepsilon) \subset f^{-1}(V)$, but then

$$f(-\varepsilon, \varepsilon) = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times \cdots\cdots \not\subset V$$

Remark. In fact

$$f^{-1}(V) = \bigcap_{k=1}^{\infty} \left(\tfrac{-1}{k}, \tfrac{1}{k}\right) = \{0\}$$