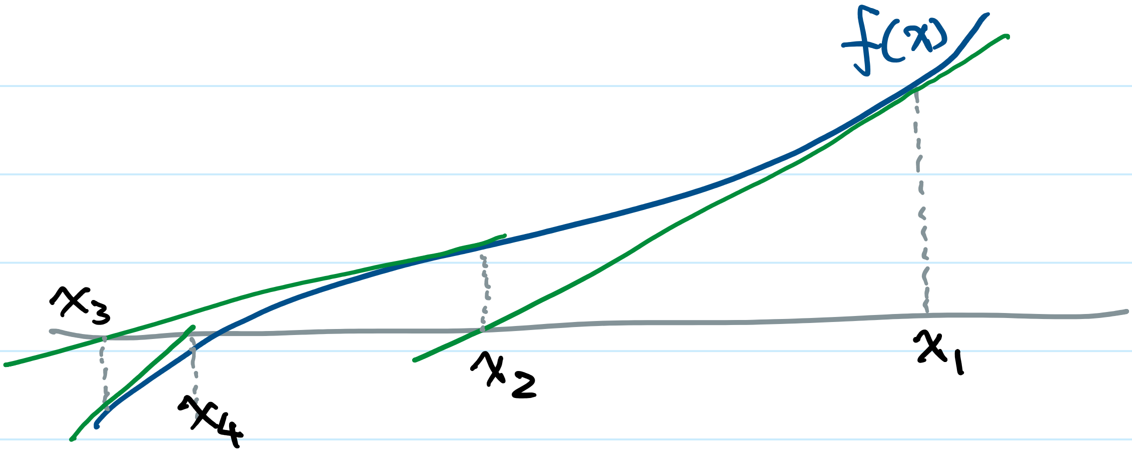
Lect09-20180205

Jan 30, Tuesday, 2018 6:43 PM

What are the major properties for Complete metric spaces ?

Every Cauchy sequence converges

Question How to prove Newton's Method



Under the condition that $f' \neq 0$ there,

$x_1, x_2, x_3, \cdots, x_n, \cdots$ is Cauchy

$\therefore x_n \longrightarrow x_\infty, f(x_\infty) = 0$ why ??

Because $\underbrace{|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|}$

Often used in computer programs

Contraction Mapping Theorem.

Given a metric space (X, d). A mapping $f: X \longrightarrow X$ is a **contraction mapping** if $\exists$ $0 < \xi < 1$ such that $\forall$ $x_1, x_2 \in X$

$$d(f(x_1), f(x_2)) < \xi \cdot d(x_1, x_2)$$

If X is complete then every contraction mapping has a fixed point, i.e.,

$$\boxed{\exists\ x_0 \in X \text{ such that } f(x_0) = x_0}.$$

Idea of proof. Start from any $x_1 \in X$

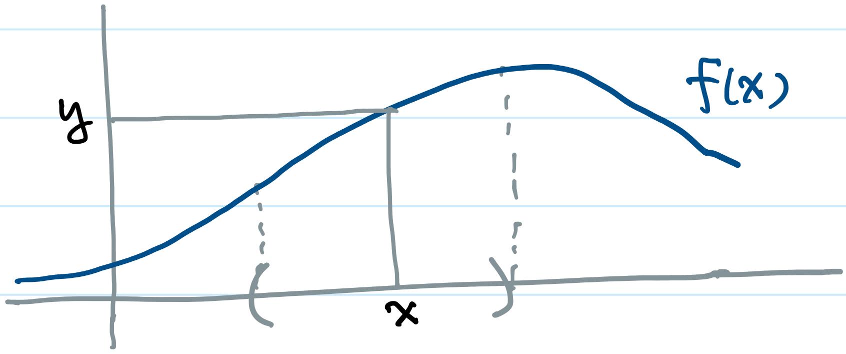
Show that $x_{n+1} = f(x_n)$ defines a Cauchy sequence.

$$d(x_{n+p}, x_n) \leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)$$

$$< \left(\xi^{n+p-2} + \xi^{n+p-3} + \cdots + \xi^{n} + \xi^{n-1}\right) d(x_2, x_1)$$

Where did we use Contraction Mapping Thm?

* Newton's Method
* Inverse / Implicit Function Theorem
* Existence of solution to ODE.

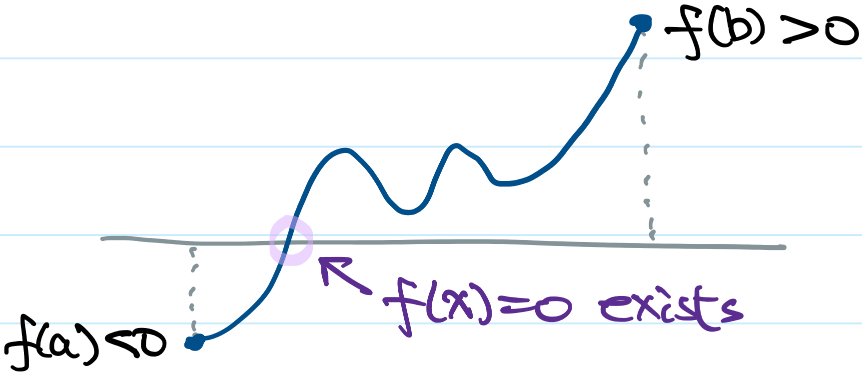


Finding inverse function $x = g(y)$

$\Longleftrightarrow$ Solving for x in $y = f(x)$ such that the solution x varies continuously as y.

i.e., Continuous Newton's Method!!

What about Intermediate Value Theorem?



How to prove it?
Use Nested Interval Theorem, which also comes from Cauchy sequence.

Cantor Intersection Theorem Let (X,d) be a complete metric space. If
- each $F_n \subset X$ is closed
- $F_{n+1} \subset F_n$ for all n
- $\operatorname{diam}(F_n) \longrightarrow 0$ as $n \to \infty$,

then $\bigcap_{n=1}^{\infty} F_n$ is a singleton $\left\{ \begin{array}{l} \text{existence,} \\ \text{uniqueness} \end{array} \right.$

Idea of proof.
1. Create a Cauchy sequence $(x_n)_{n=1}^{\infty}$

 and so $x_n \longrightarrow x$.
2. Show that $\bigcap_{n=1}^{\infty} F_n = \{x\}$.

Proof. How to get $(x_n)_{n=1}^{\infty}$?
No need to ask Ah Kwai, take any $x_n \in F_n$.

Need to show $(x_n)_{n=1}^{\infty}$ is Cauchy.

For arbitrary $\varepsilon > 0$, wish to get $N \in \mathbb{N}$, such that for $\underbrace{m > n \geq N}_{F_m \subset F_n}$, $\underbrace{d(x_m, x_n)}_{\leq \operatorname{diam}(F_n)} < \varepsilon$

$$F_m \subset F_n, \quad x_m \in F_m, \quad x_n \in F_n$$

As $\operatorname{diam}(F_n) \to 0$, $N \in \mathbb{N}$ can be obtained so that $\operatorname{diam}(F_n) < \varepsilon$.

As X is complete, $x_n \to x$ as $n \to \infty$

Now, $x \in \bigcap_{n=1}^{\infty} F_n \iff \forall\, n = 1, \cdots, \infty$, $x \in F_n$

Why?

The sequence $(x_m)_{m=n}^{\infty}$, $x_m \to x$ as $m \to \infty$

$$(x_m)_{m=n}^{\infty} \subset F_n \implies x \in \overline{F_n} = F_n$$

closed

Finally, uniqueness $\{x\} = \bigcap_{n=1}^{\infty} F_n$?

Suppose $x, y \in \bigcap_{n=1}^{\infty} F_n$ where $x \neq y$

Then $\forall\, n = 1, 2, \cdots$, $\operatorname{diam}(F_n) > d(x,y) \neq 0$

$$\therefore \lim_{n \to \infty} \operatorname{diam}(F_n) \geq d(x,y) \neq 0$$

contradiction

Revisit MATH 3060: Baire Category

Definition. A space X is of 1^{st} Category if
$$X = \bigcup_{k=1}^{\infty} N_k \quad \text{where} \quad \underbrace{(\overline{N_k})^\circ = \phi}_{\text{Definition of nowhere dense}} \quad \forall k$$

Exercise. $(\overline{N})^\circ = \phi \iff X \setminus \overline{N}$ is dense
$\iff \forall\, U \in \mathfrak{J},\ U \setminus \overline{N}$ is dense in U.

Example.
* $\mathbb{Q}$ is dense in $\mathbb{R}$ and $(\mathbb{Q}, \mathfrak{J}_{std})$ is of 1^{st} Cat.
* $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$ but $(\mathbb{Z}, \mathfrak{J}_{std})$ is of 2^{nd} Category

Baire Category Theorem Every complete metric space is of 2^{nd} category Not 1^{st} category

If (X,d) is complete then it is impossible to have $X = \bigcup_{k=1}^{\infty} N_k$ where for each k, $(\overline{N_k})^\circ = \phi$.

Proof. Suppose $X=\bigcup_{k=1}^{\infty} N_k$, $(\overline{N_k})^\circ=\emptyset$

Recall $(\overline{N})^\circ=\emptyset \iff \forall\, U\in\mathcal{T}$, $U\setminus\overline{N}$ is dense in U.

Wish

$X\setminus\overline{N_1}$ has a lot in X

$\cup$

$(X\setminus\overline{N_1})\setminus\overline{N_2}$ " " " " $X\setminus\overline{N_1}$

$\cup$

$\vdots$

$X\setminus\bigcup_{k=1}^{n}\overline{N_k}$ " " " " $X\setminus\bigcup_{k=1}^{n-1}\overline{N_k}$

$\parallel$

$$\bigcap_{k=1}^{n}(X\setminus\overline{N_k}) \supset \bigcap_{k=1}^{n} F_k$$

$\downarrow$? $\qquad\qquad$ $\downarrow$?

$\emptyset$ $\qquad\qquad$ Singleton

by given $\qquad$ by Cantor Intersection

Since $X \setminus \overline{N_1}$ is dense, pick $x_1 \in X \setminus \overline{N_1}$

$\overset{\text{open}}{\Longrightarrow} \exists\ x_1 \in B(x_1, 2r_1) \subset X \setminus \overline{N_1}$

$$\cup$$

$$F_1 = \{x \in X : d(x, x_1) \le r_1\}$$

$$\cup$$

$$B(x_1, r_1) \setminus \overline{N_2} \subset B(x_1, r_1)$$

$$\cup$$

Similarly, $\exists\ x_2 \in B(x_2, 2r_2)$

$$\cup$$

$$F_2 = \{x \in X : d(x, x_2) \le r_2\}$$

and so on, $F_1 \supset F_2 \supset F_3 \supset \cdots \supset F_n \supset$

$$\begin{array}{ccccc} \uplus & \uplus & \uplus & & \uplus \\ x_1 & x_2 & x_3 & \cdots & x_n \end{array}$$

Each $F_n \subset B(x_n, 2r_n)$, $r_{n+1} = \frac{r_n}{2}$

$\therefore$ diam $F_n \longrightarrow 0$

$$\exists\ x \in \bigcap_{k=1}^{\infty} F_k \subset X \setminus \bigcup_{k=1}^{\infty} \overline{N_k} = \phi$$

contradiction

Finite Product of $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$.

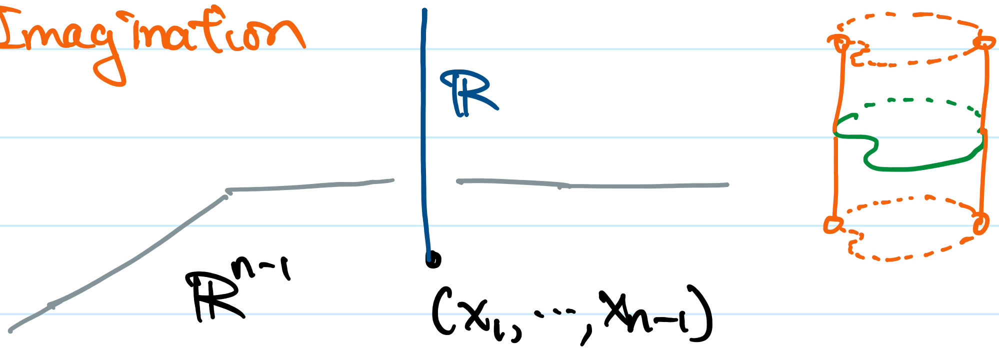
The **product topology** $\mathcal{T}_{X \times Y}$ for $X \times Y$ is generated by

$$\mathcal{S} = \{U \times Y : U \in \mathcal{T}_X\} \cup \{X \times V : V \in \mathcal{T}_Y\}$$

After taking finite intersections on $\mathcal{S}$, get a base $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$

Example. $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$

Imagination



A base $\mathcal{B} = \{U \times (a,b) : U \in \mathcal{T}_{std}\}$

Example. Annulus and Cylinder

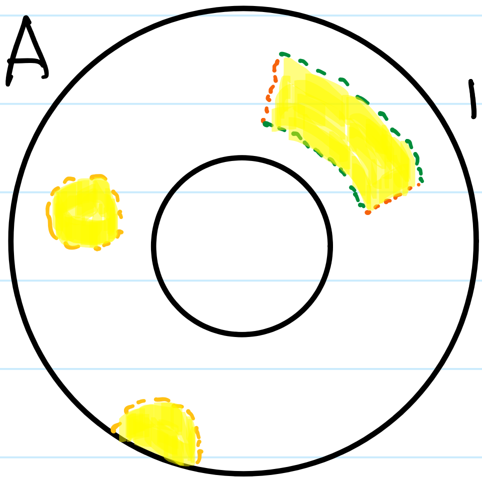
$$A = \{z \in \mathbb{C} : a \leq |z| \leq b\} \subset \mathbb{C} = \mathbb{R}^2$$

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C} = \mathbb{R}^2$$

Subspace topology

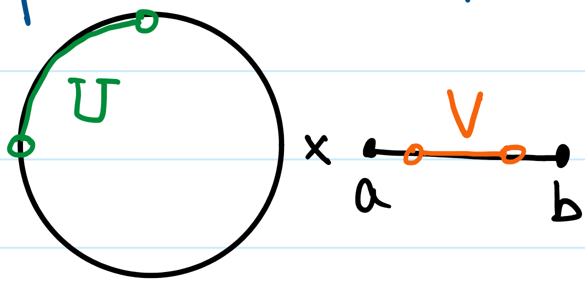
Lect09-p10

Feb 07, Wednesday, 2018 11:44 AM



$z = |z|e^{i\theta} \rightleftarrows (e^{i\theta}, |z|) \in S^1 \times [a,b]$

homeomorphic product space

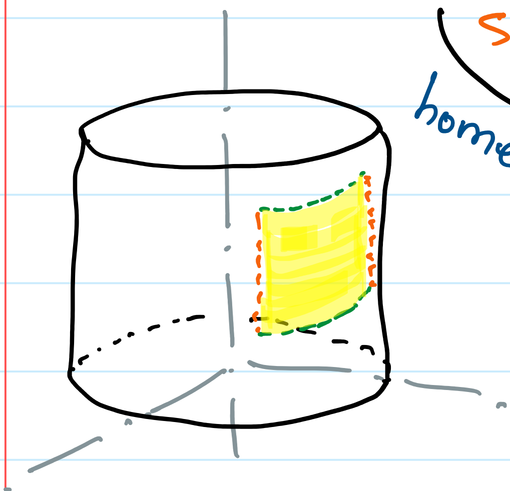


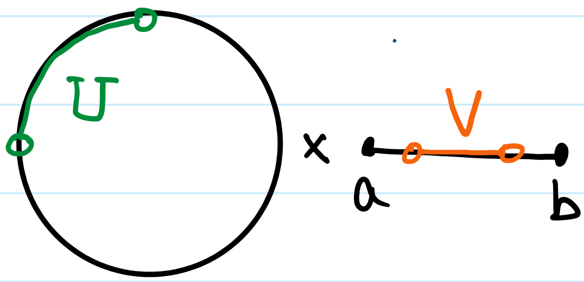
Definition. A mapping $h : X \longrightarrow Y$ is a homeomorphism if h is 1-1 & onto and both h, h^{-1} are continuous.

$CYL = \{ \underline{(u,v,w) \in \mathbb{R}^3} : u^2+v^2=1,\ w \in [a,b] \}$

subspace topology from $\mathbb{R}^3$

homeomorphic $\longrightarrow (u+iv, w) \in S^1 \times [a,b]$





Exercise. Verify the above homeomorphisms with mathematical writings.

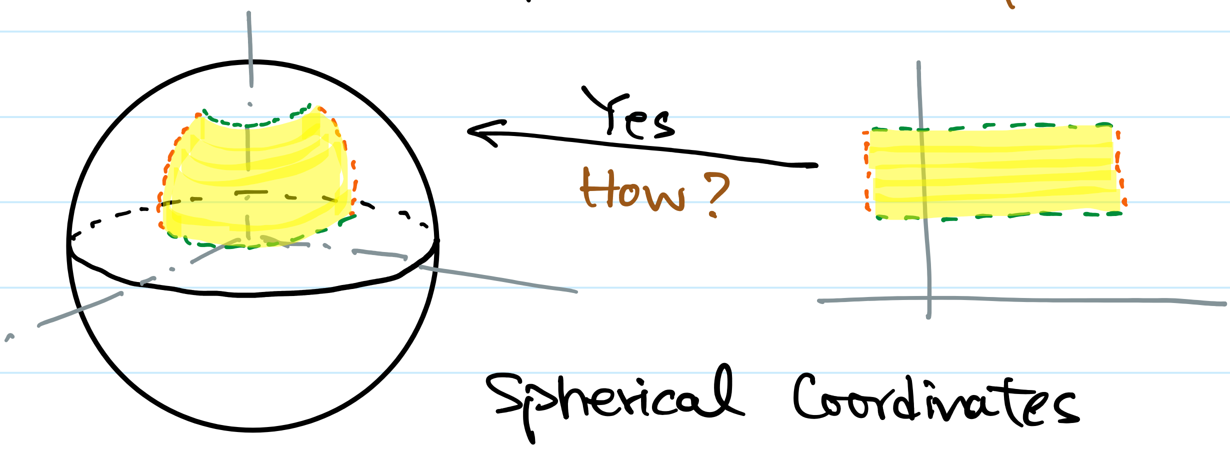
Lect09-p11

Monday, 5 February 2018 10:19 AM

n-Spheres

$$S^n = \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \} \subset \mathbb{R}^{n+1} \quad \text{subspace}$$

Can we write an open nbhd as a product?



Spherical Coordinates

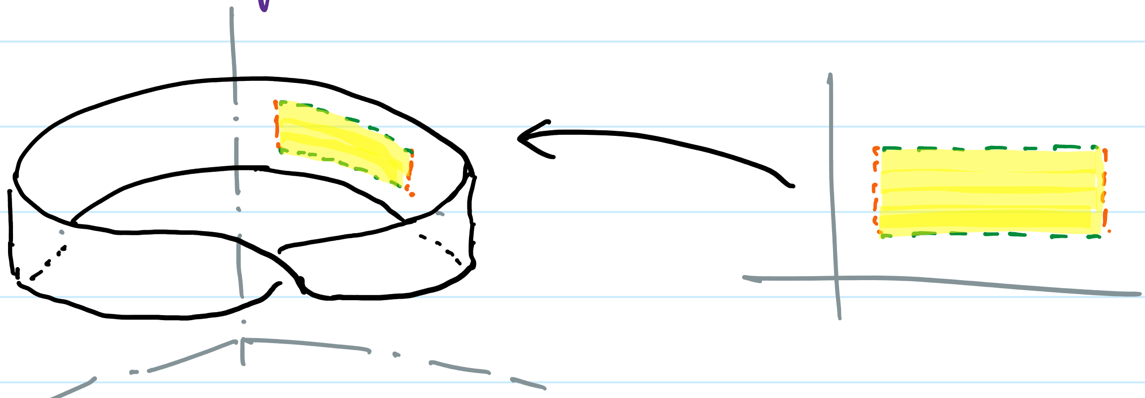
$$(\cos\phi \cos\theta, \cos\phi \sin\theta, \sin\phi) \longleftarrow\!\shortmid (\theta, \phi)$$

$$\text{Image} \subsetneq S^2$$

Locally product $\overset{\times}{\Longrightarrow}$, $\Longleftarrow$ Homeomorphic to Product

Every point has a nbhd homeomorphic to a product

Möbius Strip $\subset \mathbb{R}^3$



Exercise. Write mathematically the Möbius strip as a subset of $\mathbb{R}^3$.