Mapping $f:(X,\mathcal{T}_X) \longrightarrow (Y,\mathcal{T}_Y)$ that respects / preserves topological properties.

What is such mapping ?

Example. Continuity of $f:\mathbb{R}^n \to \mathbb{R}^n$ at $x_0 \in \mathbb{R}^n$

Write the definition please.

$\forall\ \varepsilon>0 \quad \exists\ \delta>0$ such that

$$\text{if } \underbrace{\|x-x_0\|<\delta}_{d_X(x,x_0)<\delta} \text{ then } \underbrace{\|f(x)-f(x_0)\|<\varepsilon}_{d_Y(f(x),f(x_0))<\varepsilon}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

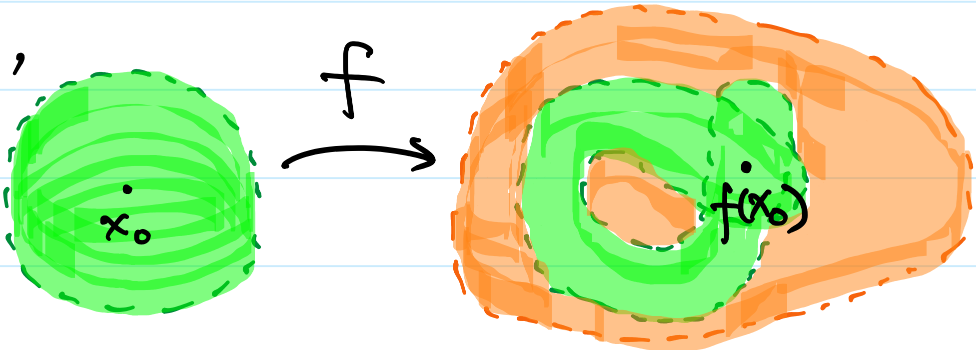
$$x \in B_X(x_0,\delta) \qquad\qquad f(x) \in B_Y(f(x_0),\varepsilon)$$

Rewrite : if $x \in U$ then $f(x) \in V$

Say it again, in set language !

$$f(U) \subset V \quad \text{or} \quad U \subset f^{-1}(V)$$

In picture,



Remember, still need to translate

$\forall\ \varepsilon>0 \quad \exists\ \delta>0 \quad \cdots\cdots$

Definition. $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ is continuous at $x_0\in X$ if

$\forall\ V\in\mathcal{T}_Y$ with $f(x_0)\in V$ nbhd of $f(x_0)$

$\exists\ U\in\mathcal{T}_X$ with $x_0\in U$ nbhd of x_0

such that $U\subset f^{-1}(V)$

Think about continuous everywhere!

$\forall\ x_0\in X\ \ \forall\ V\in\mathcal{T}_Y$ with $f(x_0)\in V$

$\exists\ U\in\mathcal{T}_X$ with $x_0\in U\subset f^{-1}(V)$

As x_0 is arbitrary, the above can be

$\forall\ V\in\mathcal{T}_Y$ and $\forall\ x_0\in f^{-1}(V)$

$\exists\ U\in\mathcal{T}_X$ with $x_0\in U\subset f^{-1}(V)$

Give me one short sentence

$$f^{-1}(V)\in\mathcal{T}_X$$

Definition. A mapping $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ is continuous (everywhere) if

$\forall\ V\in\mathcal{T}_Y$, $f^{-1}(V)\in\mathcal{T}_X$

Lect05-p3

Sunday, 21 January 2018 1:58 PM

Example. Dirichlet Function $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

* $f: (\mathbb{R}, \mathcal{J}_{std}) \longrightarrow (\mathbb{R}, \mathcal{J}_{std})$ is not continuous

$V = (\frac{-1}{2}, \frac{1}{2})$, $f^{-1}(V) = \mathbb{Q} \notin \mathcal{J}_{std}$

Only covered the case $x_0 \in \mathbb{Q}$

* It is continuous (everywhere) as

$f: (\mathbb{R}, \mathcal{P}(\mathbb{R})) \longrightarrow (\mathbb{R}, \text{any})$

certainly contains $f^{-1}(V)$ $\forall$ $V \subset \mathbb{R}$

* It is also continuous as

$f: (\mathbb{R}, \text{any}) \longrightarrow (\mathbb{R}, \{\phi, \mathbb{R}\})$

must have $\phi = f^{-1}(\phi)$, $\mathbb{R} = f^{-1}(\mathbb{R})$

Remark. Continuity of $f: (X, \mathcal{J}_X) \longrightarrow (Y, \mathcal{J}_Y)$ involves properties of $f, \mathcal{J}_X, \mathcal{J}_Y$.

Example. Let $\mathcal{X} = \{$ continuous functions $: [a,b] \to \mathbb{R}\}$

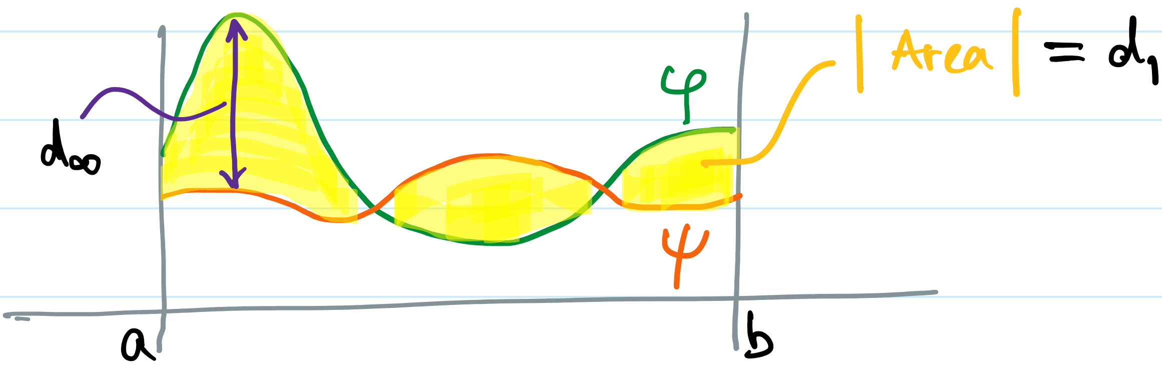
Simply consider $id : \mathcal{X} \longrightarrow \mathcal{X}$ and

L_1-Topology, $\mathcal{T}_1$, by the metric

$$d_1(\varphi, \psi) = \int_a^b |\varphi(t) - \psi(t)|\, dt$$

Uniform Topology, $\mathcal{T}_\infty$, by

$$d_\infty(\varphi, \psi) = \sup\{|\varphi(t) - \psi(t)| : t \in [a,b]\}$$



Which is TRUE ?

(a) $id : (\mathcal{X}, \mathcal{T}_\infty) \longrightarrow (\mathcal{X}, \mathcal{T}_\infty)$ is continuous

(b) $id : (\mathcal{X}, \mathcal{T}_1) \longrightarrow (\mathcal{X}, \mathcal{T}_\infty)$ is continuous

(c) $id : (\mathcal{X}, \mathcal{T}_\infty) \longrightarrow (\mathcal{X}, \mathcal{T}_1)$ is continuous

(d) None of the above

(e) All of the above

Lect05-p5

Sunday, 21 January 2018 2:53 PM

(a) is trivial

Exercise Any $id : (X, \mathcal{T}_X) \to (X, \mathcal{T}_X)$

(d) is a consequence. $\therefore$ (e) is not true

(c) is elementary

Idea. $d_\infty(\varphi, \psi) < \delta = \frac{\varepsilon}{b-a}$

$$\Rightarrow \quad d_1(\varphi, \psi) < \int_a^b \frac{\varepsilon}{b-a}\, dt = \varepsilon$$

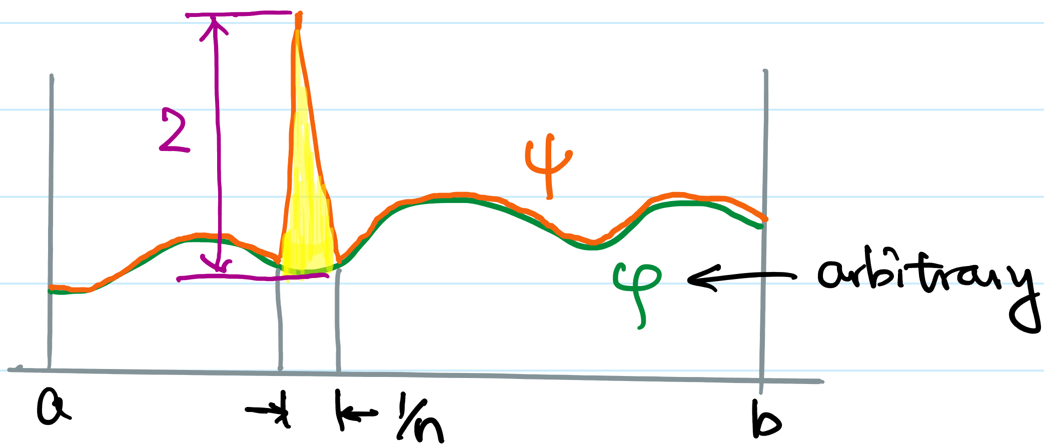
Metric Argument about $id : (\mathfrak{X}, \mathcal{T}_1) \to (\mathfrak{X}, \mathcal{T}_\infty)$

At any $\varphi \in \mathfrak{X}$, choose $\varepsilon = 1$;

for any $\delta > 0$, take $n \in \mathbb{N}$ with $\frac{1}{n} < \delta$

Construct a continuous $\psi \in \mathfrak{X}$ such that

* $\psi \equiv \varphi$ except on an interval of size $\frac{1}{n}$
* ψ differs from φ in a "triangle" on the sub-interval of length $1/n$



What do you observe from above?

* On the sub-interval of length $1/n$

$$\varepsilon=1<\psi_{max}-\varphi_{max}\leq 2=\psi_{max}-\psi_{min}$$

$$d_\infty(\psi,\varphi)=\sup|\psi-\varphi|$$

$$d_1(\psi,\varphi)\leq\frac{1}{2}(\psi_{max}-\psi_{min})\cdot\frac{1}{n}\leq\frac{1}{2}\cdot 2\cdot\frac{1}{n}<\delta$$

This proves that $id:(\mathfrak{X},\mathcal{T}_1)\longrightarrow(\mathfrak{X},\mathcal{T}_\infty)$ is not continuous at arbitrary $\varphi\in\mathfrak{X}$.

Exercise. Rewrite the above in open sets.

Equivalences of Continuity, $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ with bases $\mathcal{B}_X, \mathcal{B}_Y$ of $\mathcal{T}_X, \mathcal{T}_Y$ respectively.

① f is continuous at x $\forall x \in X$

$\Updownarrow$ known, definition

② $\forall V \in \mathcal{T}_Y,\ f^{-1}(V) \in \mathcal{T}_X$

$\Downarrow$ trivial $\Uparrow$ $f^{-1}(\text{union}(\because)) = \text{union}\cdot f^{-1}(\because)$

③ $\forall B \in \mathcal{B}_Y,\ f^{-1}(B) \in \mathcal{T}_X$ (not always in $\mathcal{B}_X$)

$\Downarrow$

④ $\forall E \subset X,\ f(\overline{E}) \subset \overline{f(E)}$ Extreme examples

$\Downarrow$ obvious, $ff^{-1}(\bullet) = \bullet$

⑤ $\forall F \subset Y,\ \overline{f^{-1}(F)} \subset f^{-1}(\overline{F})$

$\Downarrow$ Easy

⑥ $\forall$ closed $H \subset Y,\ f^{-1}(H)$ is closed in X

Interpretation of ④ $f(\overline{E})$ and $\overline{f(E)}$

* It is about two operations

$$f(\text{Closure}(\cdot)) \quad \text{or} \quad \text{Closure}(f(\cdot))$$

* Worse Discontinuous case

$$(\cdot, \{\phi, X\}) \longrightarrow (\cdot, \mathcal{P}(X))$$

For $E = \{x\}$, $\overline{E} = X$, $f(\overline{E}) = f(X)$ big

$f(E) = \{f(x)\}$, $\overline{f(E)} = \{f(x)\}$ small

In this case, $f(\overline{E}) \not\subset \overline{f(E)}$

From ④ $\Rightarrow$ ⑤ Take $E = f^{-1}(F)$, $\therefore f(E) = F$

$$f(\overline{f^{-1}(F)}) = f(\overline{E}) \subset \overline{f(E)} = \overline{F}$$

Take f^{-1}, $\overline{f^{-1}(F)} \subset f^{-1}f(\overline{f^{-1}(F)}) \subset f^{-1}(\overline{F})$

From ⑤ $\Rightarrow$ ⑥ Take $F = H$, $\therefore \overline{F} = \overline{H} = H = F$

$$\overline{f^{-1}(H)} \subset f^{-1}(\overline{H}) = f^{-1}(H),$$

② $\forall V \in \mathcal{J}_Y$, $f^{-1}(V) \in \mathcal{J}_X$

③ $\forall B \in \mathcal{B}_Y$, $f^{-1}(B) \in \mathcal{J}_X$

$\Downarrow$

④ $\forall E \subset X$, $f(\overline{E}) \subset \overline{f(E)}$

Give me your first step and wish !!

Give me your first step and wish !!

First: Let $f(x) \in f(\overline{E})$, i.e., $x \in \overline{E}$

Wish: $f(x) \in \overline{f(E)}$

Expand the statements

$\forall\, V \in \mathcal{J}_Y$ with $f(x) \in V$, $V \cap f(E) \neq \emptyset$

$\forall\, U \in \mathcal{J}_X$ with $x \in U$, $U \cap E \neq \emptyset$

Let $V \in \mathcal{J}_Y$ with $f(x) \in V$

Create $U \in \mathcal{J}_X$ with $x \in U$

By continuity of f,

$$U = f^{-1}(V) \in \mathcal{J}_X, \quad x \in f^{-1}(f(x)) \subset U$$

$$\therefore \quad \exists\, e \in U \cap E \neq \emptyset$$

Then $f(e) \in f(U \cap E) \subset f(U) \cap f(E)$

$$\subset V \cap f(E)$$

$$\therefore \quad V \cap f(E) \neq \emptyset$$