Recall the two countability about topology

C_{II} : $\exists$ countable base $\mathcal{B}$ for $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$

C_I : Every $x \in X$ has a countable local base $\mathcal{U}_x$

$\forall\, G \in \mathcal{T}$ and $x \in G$, $\exists\, U \in \mathcal{U}_x$ $x \in U \subset G$

$C_{II} \Rightarrow C_I$ because

$\mathcal{U}_x = \{B \in \mathcal{B} : x \in B\}$ is a local base

$C_{II} \cancel{\Leftarrow} C_I$ Example is Lower-limit topology

Simple Example of a C_{II} topology on an infinite set X ??

Insult my intelligence

Obviously, $\mathbb{R}^n$ with $\mathcal{T}_{std}$

$$\mathcal{B} = \left\{ B\left(q, \tfrac{1}{n}\right) : 1 \le n \in \mathbb{N},\ q \in \mathbb{Q}^n \right\}$$

What is special about $\mathbb{Q}^n$ or $\mathbb{Q}$?

What is Archimedes Property ?

$$\forall\, x, y \in \mathbb{R}\ \ \exists\, n \in \mathbb{N}\quad nx > y$$

Exercise. Show that

$\forall\ r<s\in\mathbb{R}\ \exists\ q\in\mathbb{Q}\quad r<q<s$

$\Downarrow$ because $G\supset(r,s)$ for some r,s

$\forall$ nonempty open set G in $\mathbb{R}$, $\underbrace{\exists\ q\in\mathbb{Q} \text{ and } q\in G}_{G\cap\mathbb{Q}\neq\emptyset}$

Think about the statement

$\forall$ nonempty $G\in\mathcal{T}$, $\mathbb{Q}\cap G\neq\emptyset$

What happens to arbitrary $x\in\mathbb{R}$

For any $\underbrace{\text{nbhd } N \text{ of } x}_{U\in\mathcal{T} \text{ with } x\in U}$

must have $N\cap\mathbb{Q}\neq\emptyset$

That means $x\in\overline{\mathbb{Q}}$

Since this is true for arbitrary $x\in\mathbb{R}$

i.e., $\overline{\mathbb{Q}}=\mathbb{R}$

Definition. In a topological space $(X,\mathcal{T})$
A subset $D\subset X$ is dense if $\overline{D}=X$.

A topological space is separable if $\exists$ countable subset

Question

$$\begin{tikzcd}
C_{II} \arrow[rr, Rightarrow, "\text{Yes}"] \arrow[dr, Rightarrow, "{??\ (b)}"'] & & C_{I} \arrow[dl, Rightarrow, "{??\ (a)}"] \\
& \text{Separable} &
\end{tikzcd}$$

What do you guess ?

Think about two copies of $\mathbb{R}$, mathematically, i.e., $\mathbb{R}\times\{0\} \cup \mathbb{R}\times\{1\}$, each with $\mathcal{T}_{std}$. Then a countable dense set is $\mathbb{Q}\times\{0\}\cup\mathbb{Q}\times\{1\}$.

What if we clone a separable space uncountably many ?

Can the new dense set be countable ?

Example. A singleton set $\{x\}$ is clearly separable. Clone it uncountably many copies. We get an uncountable set of discrete topology !

$C_{II} \Longrightarrow$ separable

We have a countable base
$$\mathcal{B} = \{B_k : k \in \mathbb{N}\}$$
Need to construct a countable set

How ??

Easiest Method
$$Q = \{x_k \in B_k : k \in \mathbb{N}\}$$
pick any

Need to prove $\overline{Q} = X$

What will you do ?

Try to prove something about any $x \in X$

Refer to the definition of dense

Let $U \in \mathcal{J}$ and $x \in U$ (or nbhd N of x)
$$\overset{\parallel}{\bigcup_{j\geq 1}} B_{k_j} \quad \text{as } \mathcal{B} \text{ is a base}$$
Thus, $x \in B_{k_j}$ for some k_j and $B_{k_j} \neq \emptyset$

Finally, $x_{k_j} \in B_{k_j} \subset U$ and $x_{k_j} \in Q$

Question. What question to ask ?

Assume X is C_I and separable

Let us try to show C_{II}

Known: Every $x \in X$ has countable $\mathcal{U}_x$

$\exists$ countable Q with $\overline{Q} = X$

Natural candidate of countable base $\mathcal{B}$

$$\mathcal{B} = \bigcup_{q \in Q} \mathcal{U}_q$$

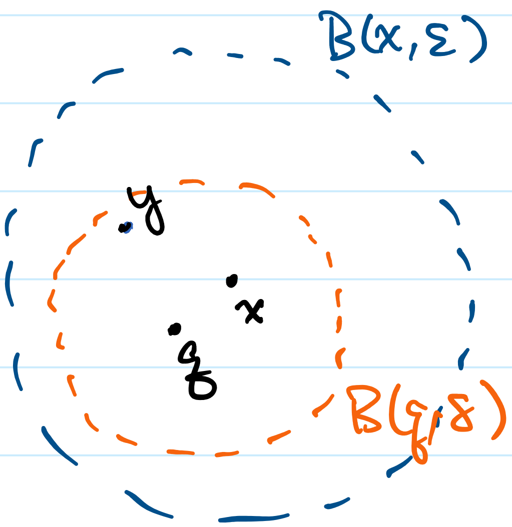
Try to repeat the argument of $\mathbb{Q}^n$ in $\mathbb{R}^n$

Take any open set $G \in \mathcal{J}$

For $x \in G$, we have $B(x,\varepsilon) \subset G$ $\quad U_x \in \mathcal{U}_x$

Need to "replace" $B(x,\varepsilon)$ by $B(q,\delta)$

U_x by $U_q \in \mathcal{U}_q$



Why is this picture possible in $\mathbb{R}^n$?

How can we make sure

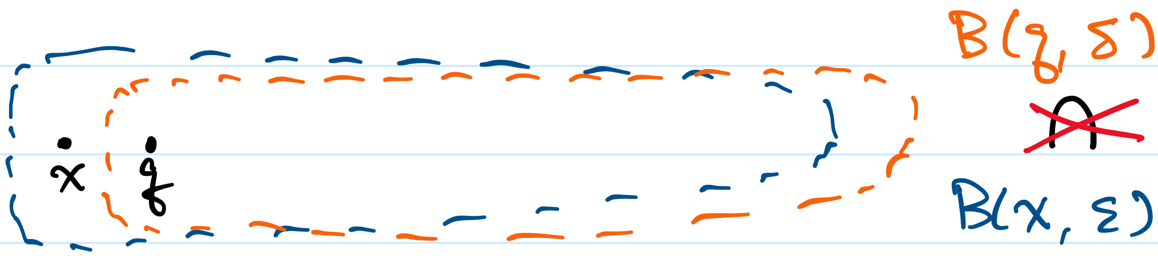
$\forall\, y \in q$, $y \in x$

It is the Δ-inequality

Without metric, bad situation may happen

Jan 18, Thursday, 2018 2:15 PM

Bad Situation



Example. For $(\mathbb{R}, \mathcal{T}_{ll})$, lower-limit topology

It is C_I.

$$\mathcal{U}_x = \{[x, x+\tfrac{1}{n}) : 1 \le n \in \mathbb{N}\}$$

It is separable

$\mathbb{Q}$ is countable and still dense in this topology

It is not C_{II}

We mentioned before.