Lect02-20180110

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In the context of metric spaces, recall the definition of interior.

$A \subset X$ and $x_0 \in A$ is an interior point if $\exists\ \varepsilon > 0$

$$x_0 \in B(x_0, \varepsilon) \subset A$$

Without metric, only $(X, \mathcal{T})$, how to define this concept?

Definition. Let $A \subset X$.
$x_0 \in A$ is an interior point of A
if $\exists\ U \in \mathcal{T}$, $x_0 \in U \subset A$
Denoted $x_0 \in \mathring{A}$ or $\mathrm{Int}(A)$

Definition. Let $x \in X$.
$N \subset X$ is a neighborhood of x
if $x \in \mathring{N}$

note: may not be open.

Exercise. Use the above definition, prove

$$A \in \mathcal{T} \iff A = \mathring{A}$$

Question. Given $(X, \mathcal{T})$ and $x \in X$.
Write $\mathcal{N}_x = \{ N \subset X : x \in \mathring{N} \}$

List some conditions that $\mathcal{N}_x$ satisfies

Hints.

(N1) Let $N \in \mathcal{N}_x$, relation between x and N?

Answer: $x \in N \;\forall\; N \in \mathcal{N}_x$

(N2) Let $M, N \in \mathcal{N}_x$, what can you say?

Answer: $M \cap N \in \mathcal{N}_x$

Note: (N2) $\Longrightarrow$ $\mathcal{N}_x$ is closed under finite $\cap$.

(N3) Let $N \in \mathcal{N}_x$ and $N \subset M$,

what happens to M?

Answer. $M \in \mathcal{N}_x$

Note: (N3) $\Longrightarrow$??

$\mathcal{N}_x$ is closed under arbitrary union

(N4) Let $A \subset X$, consider the set $\{y \in A : A \in \mathcal{N}_y\}$

What is it?

Answer: With $\mathcal{J}$ given, it is $\mathring{A}$

Moreover, if we know $A \in \mathcal{N}_x$

what about the above set?

$A \in \mathcal{N}_x \Longrightarrow \{y \in A : A \in \mathcal{N}_y\} \in \mathcal{N}_x$

Note: This last fact implies $(\mathring{A})^\circ = \mathring{A}$.

Now, we can see the relation between topology and neighborhood system.

Theorem. Suppose a set X has a mapping $x \longmapsto \mathcal{N}_x \subset \mathcal{P}(X)$ satisfying

(N1) $\forall\, N \in \mathcal{N}_x$, $x \in N$

(N2) $\forall\, M, N \in \mathcal{N}_x$, $M \cap N \in \mathcal{N}_x$

(N3) If $N \in \mathcal{N}_x$ and $N \subset M \subset X$ then $M \in \mathcal{N}_x$

(N4) Denote $A^I = \{ y \in A : A \in \mathcal{N}_y \}$

If $A \in \mathcal{N}_x$ for any x then $A^I \in \mathcal{N}_x$

Then there is a unique topology $\mathcal{T}$ for X such that $\mathcal{N}_x$ contains exactly all nbhds of x; $A^I = A^\circ$ $\forall\, A \subset X$.