Lect01-20180108

Monday, January 8, 2018 10:49 PM

One important goal of topology:

To study "limit, convergence, approximation"

Example. In the context of sequence convergence in a metric space,

$x_n \to x$ in (X, d) as $n \to \infty$

if $\forall \varepsilon > 0$, ... $\underbrace{d(x_n, x) < \varepsilon}_{x_n \in B(x;\varepsilon)}$

open ball with center x and radius ε.

Why do we want to discard metric?
How should it be done naturally?

Example ①

For a metric space (X, d), we have $d(x, y)$ for points $x, y \in X$.

But, it is difficult to have natural $d(A, B)$ for $A, B \subset X$

Example ② l_p-metric on $\mathbb{R}^2$, $1 \leq p \in \mathbb{R}$

$$d_p(\underset{(x_1,x_2)}{x}, \underset{(y_1,y_2)}{y}) = \left(|x_1-y_1|^p + |x_2-y_2|^p\right)^{1/p}$$

or

$$\max\{|x_1-y_1|, |x_2-y_2|\}, \ p=\infty$$

Exercise (done in MATH 3060)

Let $(x_n)_{n\in\mathbb{N}}$, x both in $\mathbb{R}^2$.

$\forall \ 1 \leq p, q \in \mathbb{R} \cup \{\infty\}$,

$x_n \longrightarrow x$ in l_p-metric

$\Longleftrightarrow$ $x_n \longrightarrow x$ in l_q-metric

Main Idea (shown by $p=1, q=\infty$)

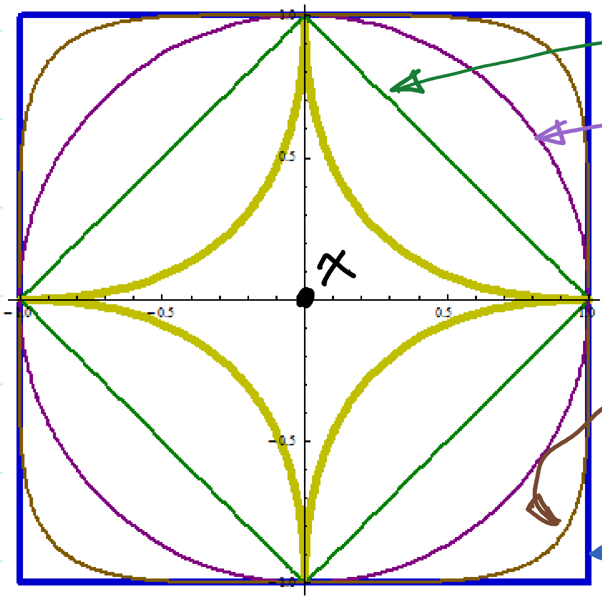
$x_n \longrightarrow x$ in l_1-metric

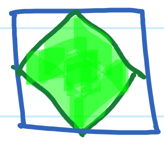
$\Longleftrightarrow \forall \varepsilon > 0 \ \cdots\cdots \ d_1(x_n, x) < \varepsilon$

$$\frac{1}{2} d_1(x_n, x) \leq d_\infty(x_n, x)$$

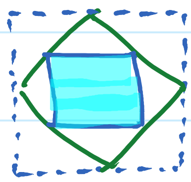
$\Longleftrightarrow \forall \varepsilon > 0 \ \cdots\cdots, \ d_\infty(x_n, x) < \varepsilon$

Let us look at pictures

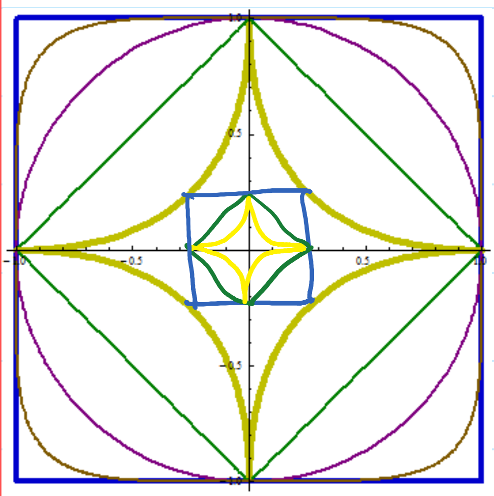


$d_1(x_n, x) < \varepsilon \iff x_n \in$ 

$\Rightarrow d_\infty(x_n, x) < \varepsilon$

$d_\infty(x_n, x) < \frac{\varepsilon}{2} \iff x_n \in$ 

$\Rightarrow d_1(x_n, x) < \varepsilon$



Convergence argument still work for without Δ-inequality

Conclusion. To discuss $x_n \to x$, metric is not essential.

Instead,

* Neighborhoods of x are important
* More precise, we need a neighborhood system at each $x \in X$

We will mention it later.

A more "elegant" approach is used.

Definition. Let X be a nonempty set. A set $\mathcal{T} \subset \mathcal{P}(X)$ is a topology for X if it satisfies the following

(T1) The union of any sets in $\mathcal{T}$ is still in $\mathcal{T}$

(T1') $\phi \in \mathcal{T}$

(T2) The intersection of any finite number of sets in $\mathcal{T}$ is still in $\mathcal{T}$.

(T2') $X \in \mathcal{T}$

$\boxed{T1+T2}$ $\mathcal{T}$ is closed under arbitrary union and finite intersection

We say that a subset $G \subset X$ is an open set wrt $\mathcal{J}$ if $G \in \mathcal{J}$.

Mathematical Expression

(T1) $\forall\, \mathcal{G} \subset \mathcal{J}, \ \cup \mathcal{G} \in \mathcal{J}$

arbitrary collection of sets in $\mathcal{J}$

Take the union, which is a subset of X

Take $\mathcal{G} = \phi$, $\cup \mathcal{G} = \phi \in \mathcal{J}$ (T1')

OR: $\forall$ family $\{G_\alpha\}_{\alpha \in I}$ of $\mathcal{J}$, $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{J}$

(T2) $\forall$ finite $\mathcal{F} \subset \mathcal{J}$, $\cap \mathcal{F} \in \mathcal{J}$

finite number of sets in $\mathcal{J}$

Take their common intersection

Take $\mathcal{F} = \phi$, $\cup \mathcal{F} = X \in \mathcal{J}$ (T2')

vacuously true

OR: $\forall \{G_1, G_2, \cdots, G_n\}$ of $\mathcal{J}$, $\bigcap_{k=1}^{n} G_k \in \mathcal{J}$.

Example ① Metric Topology

Given (X, d)

Open balls $B(x,\varepsilon)$, $x \in X$, $\varepsilon > 0$ ---> Define interior

$x \in \mathring{A}$ if $x \in B(x,\varepsilon) \subset A$ for some $\varepsilon > 0$

$$\left\{ A \subset X : A \text{ is a union of open balls} \right\} = \mathcal{T} = \{ A \subset X : A = \mathring{A} \}$$

Example ② Discrete Topology $= \wp(X)$

I.e., every subset of X is open.

It comes from the discrete metric.

Reason. Need to show $A = \mathring{A}$ $\forall A \subset X$

For arbitrary $x_0 \in A$

$$x_0 \in B(x_0, \tfrac{1}{2}) = \{x_0\} \subset A$$

$$\therefore x_0 \in \mathring{A}$$

Example (3) Indiscrete Topology $= \{\phi, X\}$

Clearly, (T1) (T2) are valid.

Example (4) Cofinite Topology

$\mathcal{J} = \{\phi\} \cup \{A \subset X : X \setminus A \text{ is finite}\}$

For (T1), consider $\{G_\alpha\}_{\alpha \in I}$ in $\mathcal{J}$

$$X \setminus \bigcup_{\alpha \in I} G_\alpha = \bigcap_{\alpha \in I} \underbrace{(X \setminus G_\alpha)}_{\text{each is finite}}$$

For (T2), for $\{G_1, G_2, \cdots, G_n\}$ in $\mathcal{J}$,

$$X \setminus \bigcap_{k=1}^{n} G_k = \bigcup_{k=1}^{n} (X \setminus G_k)$$

Example (5). Co-countable Topology

Exercises

* Cofinite topology of a finite set
* Co-countable topology of a countable set
* Read the many many examples of topology in the textbook.

Practice ①

Let $X=\mathbb{R}$. Is this a topology?

$$\{\phi,\mathbb{R}\}\cup\{(a-\varepsilon,a+\varepsilon):a\in\mathbb{R},\ \varepsilon>0\}$$

Practice ②

Let $X=\mathbb{R}$ and

$$\mathcal{T}=\{\phi,\mathbb{R},[1,3],[2,4],[1,4],[2,3]\}$$

Is the set $[1,3]$ an open set?

Answer ①

No. (T1) is not satisfied.

Answer ②

Since $\mathcal{T}$ is a topology, yes, wrt $\mathcal{T}$

No, wrt standard topology of $\mathbb{R}$.

Question. We know that the Discrete Topology $\mathcal{P}(X)$ comes from a metric.

What about the Indiscrete Topology $\{\phi,X\}$?

Special Feature of metric spaces

Given (X,d) and $x \neq y \in X$

Then $d(x,y) = r > 0$ and $\frac{r}{3} > 0$

We have $x \in B(x,\frac{r}{3})$, $y \in B(y,\frac{r}{3})$

and $B(x,\frac{r}{3}) \cap B(y,\frac{r}{3}) = \phi$

need Δ-inequality

Definition. A topological space $(X,\mathcal{T})$ is Hausdorff or T_2 if $\forall\ x \neq y \in X$ $\exists\ U,V \in \mathcal{T}$, $x \in U$, $y \in V$, and

$$U \cap V = \phi$$

Fact. Every metric space is Hausdorff.

Easy to see:

Indiscrete Topology with $\#X \geq 2$ and

$\mathcal{T} = \{\phi, \mathbb{R}, [1,3], [2,4], [1,4], [2,3]\}$

both are not Hausdorff.

Hence cannot come from a metric.

Question.

Is cofinite topology Hausdorff?