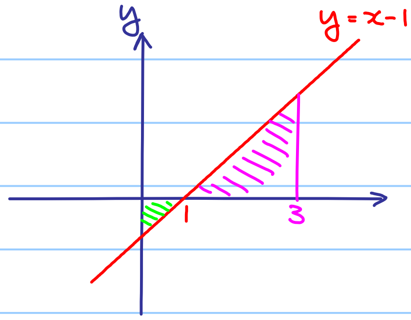
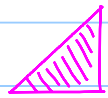
e.g. (NOT area, but signed area)




$$\int_0^1 x-1\,dx = \left[\frac{x^2}{2}-x\right]_0^1 = -\frac{1}{2}$$

$$\int_1^3 x-1\,dx = \left[\frac{x^2}{2}-x\right]_1^3 = 2$$

$$\int_0^3 x-1\,dx = \left[\frac{x^2}{2}-x\right]_0^3 = \frac{3}{2}$$

 −  (Cancellation)

e.g. $\int_{-2}^{3} |x-1|\,dx$

Recall : We can rewrite $|x-1| = \begin{cases} x-1 & \text{if } x \geqslant 1 \\ -(x-1) & \text{if } x < -1 \end{cases}$

$$\int_{-2}^{3} |x-1|\,dx = \int_{-2}^{1} |x-1|\,dx + \int_1^3 |x-1|\,dx$$

$$= \int_{-2}^{1} -(x-1)\,dx + \int_1^3 x-1\,dx$$

$\vdots$

Ex: $= \frac{9}{2} + 2 = \frac{13}{2}$

e.g. (Fundamental Theorem of Calculus)

Find $\frac{dF}{dx}$ if

a) $F(x) = \int_0^x e^{\cos t}\,dt$ , b) $F(x) = \int_0^{x^2} e^{\cos t}\,dt$ , c) $F(x) = \int_x^{x^2} e^{\cos t}\,dt$

a) $\frac{dF}{dx} = e^{\cos x}$ (Directly from Fundamental Theorem of Calculus, $f(x) = e^{\cos x}$)

b) $\frac{dF}{dx} = \frac{d}{dx^2}\int_0^{x^2} e^{\cos t}\,dt \cdot \frac{dx^2}{dx}$ (Chain rule)

$= e^{\cos x^2} \cdot 2x$

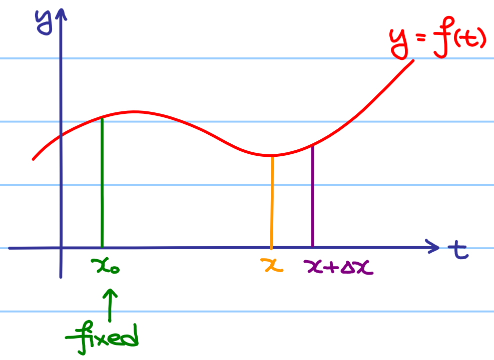
$= 2x e^{\cos x^2}$

c) $\frac{dF}{dx} = \frac{d}{dx}\int_0^{x^2} e^{\cos t}\,dt - \frac{d}{dx}\int_0^{x} e^{\cos t}\,dt$

$= 2xe^{\cos x^2} - \cos x$

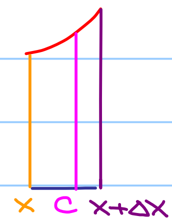
## Sketch of the proof Fundamental Theorem of Calculus

Claim: If $F(x) = \int_{x_0}^{x} f(t)\,dt$ , $\lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x} = f(x)$ , i.e. $F'(x) = f(x)$



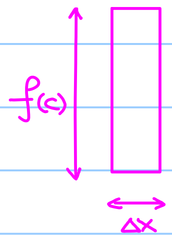

$F(x+\Delta x) - F(x)$

= Area of




By continuity of $f$, there exists $c \in (x, x+\Delta x)$ such that

=




$\lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{f(c)\,\Delta x}{\Delta x}$

$= \lim_{\Delta x \to 0} f(c)$

$= \lim_{c \to x} f(c)$ (As $\Delta x$ tends to 0, $c$ tends to $x$ )

$= f(x)$ (By continuity of $f$ )

e.g. Find $\lim_{n\to\infty} \underbrace{\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3}}_{n \text{ terms}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$

Note: As $n \to \infty$, it is an infinite sum, i.e. summing infinitely many terms.

Algebraic rule does NOT work !!

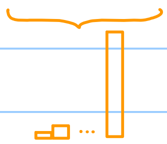
We cannot say: $\lim_{n\to\infty} \frac{1^2}{n^3} = \lim_{n\to\infty} \frac{2^2}{n^3} = \cdots = \lim_{n\to\infty} \frac{n^2}{n^3} = 0$

$\therefore \lim_{n\to\infty} \frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3} = 0$

Recall: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+(b-a)\frac{i}{n}) \cdot \frac{b-a}{n} = \int_a^b f(x)\,dx$

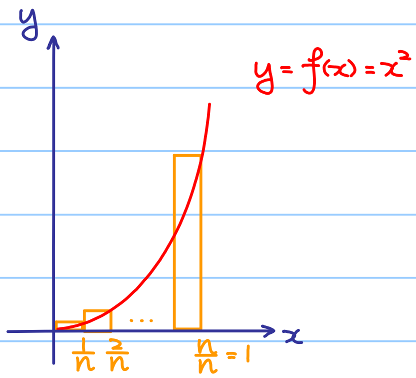
$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^2} \frac{1}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(\frac{i}{n}) \frac{1}{n}$



In this case,

$a = 0$, $b = 1$,




$= \int_0^1 f(x)\,dx$

$= \frac{1}{3}$

Roughly, $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

$\Downarrow$

$\int_a^b f(x)\,dx$

e.g. Find $\lim_{n\to\infty} \frac{1}{n}(e^{1/n}+e^{2/n}+e^{3/n}+\cdots+e^{n/n}) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{i/n}$

$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{i/n} = \lim_{n\to\infty} \sum_{i=1}^{n} e^{i/n} \cdot \frac{1}{n}$

$= \int_0^1 e^x\,dx$

$= [e^x]_0^1$

$= e^1 - e^0$

$= e - 1$

# Definite Integral Using Substitution

$$\int_a^b f(u(x))\cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

e.g. $\int_0^1 8x(x^2+1)\,dx$

$= \int_0^1 8(x^2+1)\quad x\,dx$

$= \int_1^2 8u \quad \frac{1}{2}du$

Caution !

$= \int_1^2 4u\,du$

$= [2u^2]_1^2$

$= 6$

let $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

Similar to indefinite integration

when $x=0$, $u=1$

$x=1$, $u=2$

New ! Don't forget !

Remark:

Some may write :

Still 0 and 1

$\int_0^1 8x(x^2+1)\,dx = \int_0^1 4(x^2+1)\,d(x^2+1)$

$= [2(x^2+1)]_0^1$

$= 6$

(as $d(x^2+1) = 2x\,dx$ )

(Just the same result ! )

e.g. $\int_e^{e^2} \frac{1}{x\ln x}\,dx$

$= \int_1^2 \frac{1}{u}\,du$

$= [\ln u]_1^2$

$= \ln 2 - \cancel{\ln 1}^{\,0}$

$= \ln 2$

Let $u = \ln x$

$du = \frac{1}{x}dx$

when $x=e$, $u=1$

$x=e^2$, $u=2$

## More on Subsitution

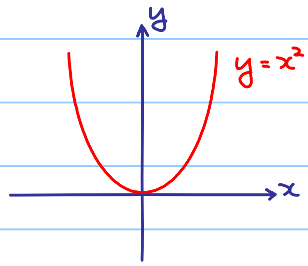
Recall :

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be

- even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$

  e.g. $x^2$, $\cos x$, $|x|$

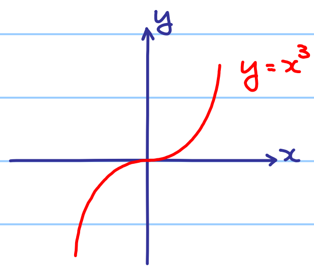
  property : the graph is symmetric along y-axis.




- odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$

  e.g. $x^3$, $\sin x$

  property : the graph is symmetric about the origin




- periodic if there exists $T > 0$ such that $f(x) = f(x+T)$ for all $x \in \mathbb{R}$

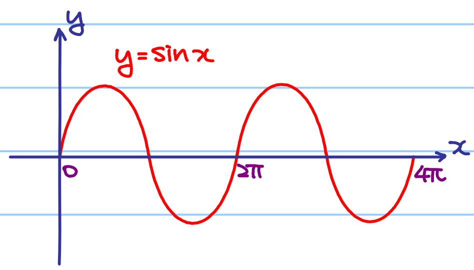
  If $T > 0$ is the least positive real number with the above property, T is called the period.

  e.g. $\sin x$, $\cos x$, $\tan x$

  property : the graph is repeating again and again

  period of $\sin x$, $\cos x$ $= 2\pi$

  period of $\tan x = \pi$

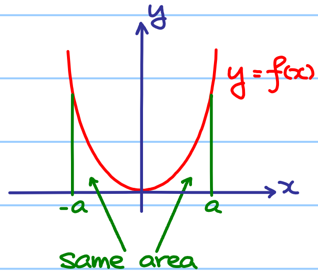


Suppose $f$ is an even function and $a>0$, prove that $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$\parallel ?$

$\int_0^a f(x)\,dx$




$$\int_{-a}^{0} f(x)\,dx \qquad \text{Let } y=-x$$

$$= \int_{a}^{0} -f(y)\,dy \qquad dy = -dx$$

$f(-y) = f(y)$ $\qquad$ When $x=0,\ y=0$

$$= \int_0^a f(y)\,dy \qquad x=-a,\ y=a$$

$$= \int_0^a f(x)\,dx \quad \text{(dummy variable)}$$

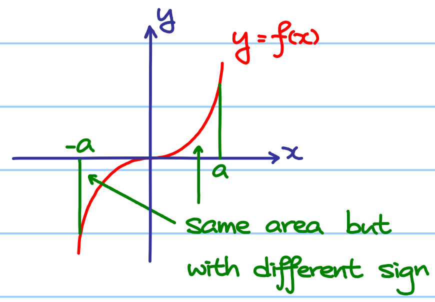
e.g. $\int_{-4}^{4} |x|\,dx = 2\int_0^4 |x|\,dx = 2\int_0^4 x\,dx = 2\left[\frac{x^2}{2}\right]_0^4 = 16$

Suppose $f$ is an odd function and $a>0$, prove that $\int_{-a}^{a} f(x)\,dx = 0$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$\parallel ?$

$-\int_0^a f(x)\,dx$




$$\int_{-a}^{0} f(x)\,dx \qquad \text{Let } y=-x$$

$$= \int_{a}^{0} -f(-y)\,dy \qquad dy = -dx$$

$f(-y) = -f(y)$ $\qquad$ When $x=0,\ y=0$

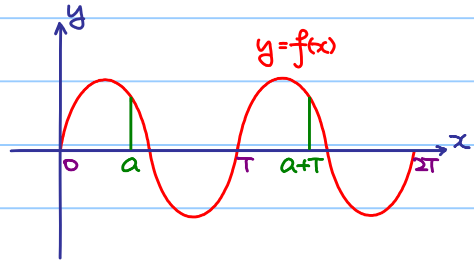
$$= \int_0^a -f(y)\,dy \qquad x=-a,\ y=a$$

$$= -\int_0^a f(x)\,dx \quad \text{(dummy variable)}$$

e.g. $\int_{-\pi}^{\pi} e^{\sin x}\,dx = 0 \quad \because e^{\sin x}$ is an odd function.

Suppose $f$ is a periodic function with period $T>0$ and $a \in \mathbb{R}$,

prove that $\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$




$$\int_a^{a+T} f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^T f(x)\,dx + \int_T^{a+T} f(x)\,dx$$

$$\underset{\displaystyle -\int_0^a f(x)\,dx}{\parallel} \qquad\qquad \underset{\displaystyle \int_0^a f(x)\,dx}{\parallel\,?}$$

$$\int_T^{a+T} f(x)\,dx$$

Let $y = x - T$

$dy = dx$

When $x = T$, $y = 0$

$x = a+T$, $y = a$

$$= \int_0^a f(y+T)\,dy$$

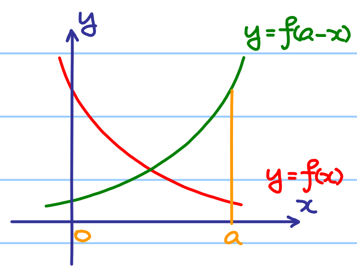
$f(y+T) = f(y)$

$$= \int_0^a f(y)\,dy$$

$$= \int_0^a f(x)\,dx \quad \text{(dummy variable)}$$

e.g. (Similar example)

Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$




$$\int_0^a f(a-x)\,dx$$

Let $y = a - x$

$dy = -dx$

When $x = 0$, $y = a$

$x = a$, $y = 0$

$$= \int_a^0 -f(y)\,dy$$

$$= \int_0^a f(y)\,dy$$

$$= \int_0^a f(x)\,dx \quad \text{(dummy variable)}$$

# Definite Integration Using Integration by Parts

$$\int_a^b u \frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v \frac{du}{dx}\,dx$$

e.g. $\int_1^e x \ln x \, dx = \int_1^e \ln x \; d\left(\frac{x^2}{2}\right)$

$$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\, d\ln x$$

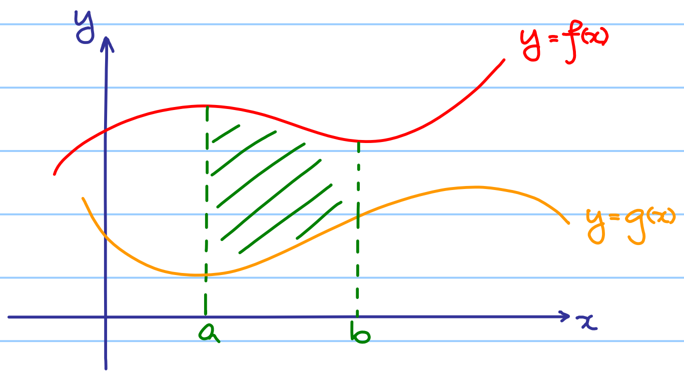
$$= \left(\frac{e^2}{2}\ln e - \underbrace{\cancel{\tfrac{1}{2}\ln 1}}_{0}\right) - \int_1^e \frac{x}{2}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$$

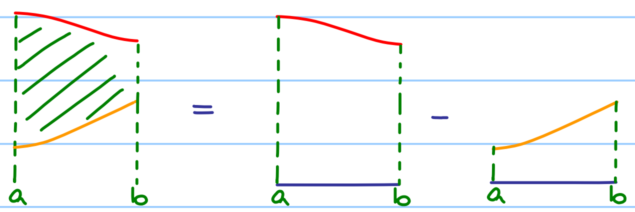
$$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

## Area Between Curves :




Area of shaded region $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$



e.g. Find the area bounded by $y=x^2$ and $y=x^3$.

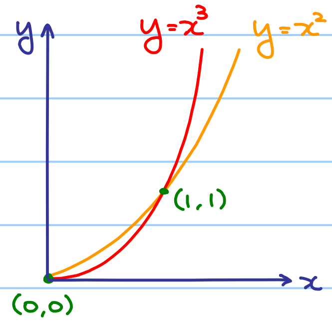
Step 1 : Solve $\begin{cases} y=x^2 \\ y=x^3 \end{cases}$

$$x^3=x^2$$
$$x^2(x-1)=0$$
$$x=0 \quad \text{or} \quad 1$$

(Remark : No need to solve $y$)

Step 2 : Note when $0 \le x \le 1$, $x^3 \le x^2$

$$\text{Area} = \int_0^1 x^2 - x^3 \, dx$$
$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$
$$= \frac{1}{12}$$




e.g. Find the area bounded by

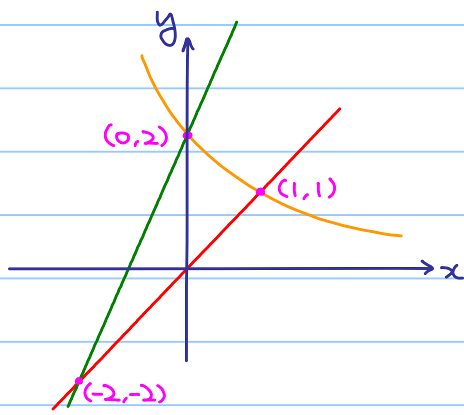
$y=f(x)=x$ , $y=g(x)=\frac{2}{x+1}$ and $y=h(x)=2x+2$

$$\text{Area} = \int_{-2}^{0} h(x) - f(x) \, dx + \int_0^1 g(x) - f(x) \, dx$$

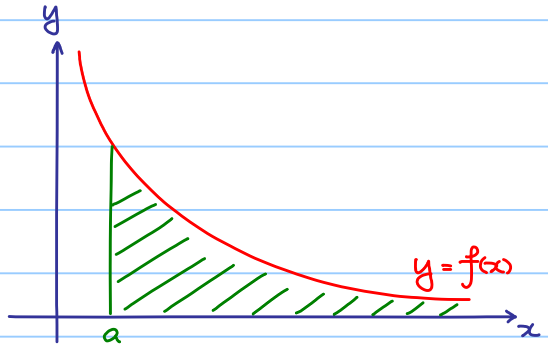
Ex : ⋮

Ans : $= 2 + (-\frac{1}{2} + \ln 4)$

$= \frac{3}{2} + \ln 4$

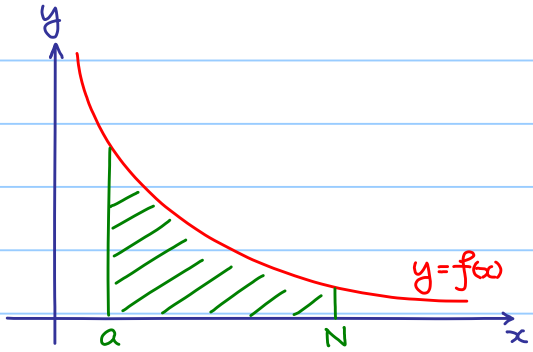


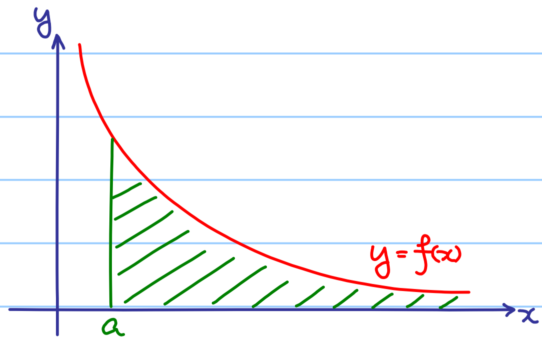
## Improper Integrals :




Question : Find the area of the unbounded region ?

Idea :



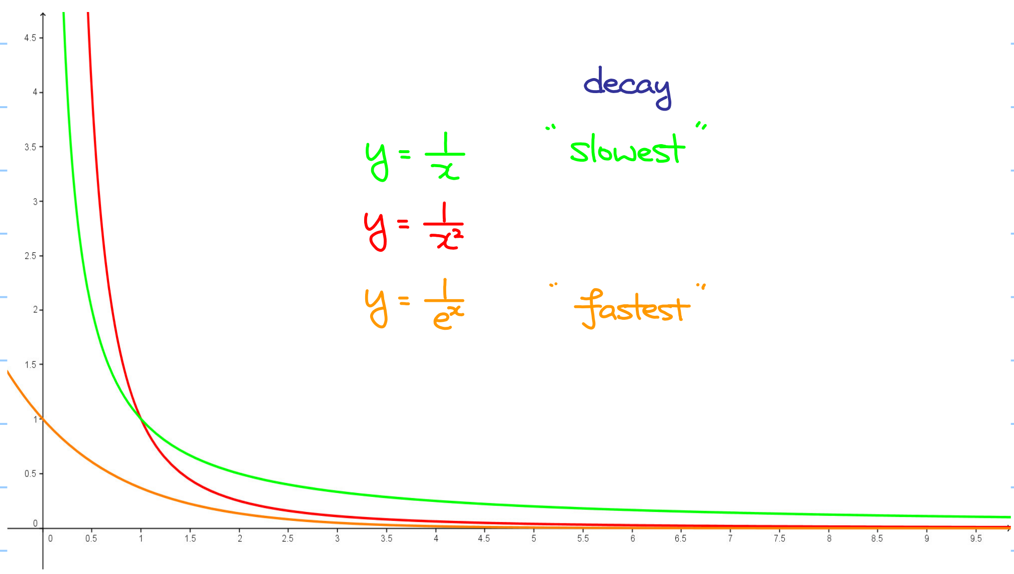




$\int_a^N f(x)\,dx \quad \rightsquigarrow \quad$ Area of the unbounded region

$= \lim_{N \to +\infty} \int_a^N f(x)\,dx \qquad$ (if it exists)

We denote it by $\int_a^{+\infty} f(x)\,dx$

e.g.



① $\lim_{N\to+\infty} \int_1^N \frac{1}{x}\,dx = \lim_{N\to+\infty} [\ln x]_1^N = \lim_{N\to+\infty} \ln N = +\infty$ (i.e. limit does NOT exist)

② $\lim_{N\to+\infty} \int_1^N \frac{1}{x^2}\,dx = \lim_{N\to+\infty} [-\frac{1}{x}]_1^N = \lim_{N\to+\infty} 1-\frac{1}{N} = 1$

③ $\lim_{N\to+\infty} \int_1^N \frac{1}{e^x}\,dx = \lim_{N\to+\infty} [-\frac{1}{e^x}]_1^N = \lim_{N\to+\infty} -\frac{1}{e^N}+\frac{1}{e}$

Observation: $\lim_{x\to+\infty} f(x)=0$ does NOT guarantee $\lim_{N\to+\infty}\int_a^N f(x)\,dx$ exists.

e.g. Find $\int_0^{+\infty} \frac{1}{(x+1)(3x+2)}\,dx$

Note: $(x+1)(3x+2)$ is a polynomial of degree 2.

$\frac{1}{(x+1)(3x+2)}$ decays as "fast" as $\frac{1}{x^2}$.

$$\lim_{N\to+\infty}\int_0^N \frac{1}{(x+1)(3x+2)}\,dx = \lim_{N\to+\infty}\int_0^N \frac{-1}{x+1}+\frac{3}{3x+2}\,dx$$

$$= \lim_{N\to+\infty} [-\ln|x+1|+\ln|3x+2|]_0^N$$

$$= \lim_{N\to+\infty} \ln\left|\frac{3N+2}{N+1}\right| - \ln 2$$

$$= \ln 3 - \ln 2$$

e.g. Find $\int_0^{+\infty} xe^{-2x}\,dx$

$$\lim_{N\to+\infty}\int_0^N xe^{-2x}\,dx$$

$$= \lim_{N\to+\infty}\int_0^N x\,d(-\tfrac{1}{2}e^{-2x})$$

$$= \lim_{N\to+\infty} [-\tfrac{1}{2}xe^{-2x}]_0^N - \int_0^N -\tfrac{1}{2}e^{-2x}\,dx$$

$$= \lim_{N\to+\infty} [-\tfrac{1}{2}xe^{-2x}]_0^N + [-\tfrac{1}{4}e^{-2x}]_0^N$$

go to 0 when $N\to+\infty$

$$= \lim_{N\to+\infty} -\tfrac{1}{2}Ne^{-2N} - \tfrac{1}{4}e^{-2N} + \tfrac{1}{4}$$

$$= \tfrac{1}{4}$$