Integration of Irrational Functions :

- Integrand with $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ $\quad (a>0)$

(1) For $\sqrt{a^2-x^2}$, we let $x = a\sin\theta \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

(2) For $\sqrt{a^2+x^2}$, we let $x = a\tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(3) For $\sqrt{x^2-a^2}$, we let $x = a\sec\theta \qquad 0 \leq \theta \leq \pi$

e.g. $\int x^2\sqrt{4-x^2}\,dx$

$$
\begin{aligned}
&\int x^3\sqrt{4-x^2}\,dx \\
=&\int 8\sin^3\theta\sqrt{4\cos^2\theta}\,(2\cos\theta)\,d\theta \\
=&\int 32\cos^2\theta\sin^3\theta\,d\theta \\
=&\int 32\cos^2\theta\,\sin^2\theta\,\sin\theta\,d\theta \\
=&\int 32\cos^2\theta\,(1-\cos^2\theta)\,d(-\cos\theta) \\
=&\int 32\cos^4\theta - 32\cos\theta\,d\cos\theta \\
=&\frac{32}{5}\cos^5\theta - \frac{32}{3}\cos^3\theta + C \\
=&\frac{32}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C \\
=&-\frac{1}{15}(3x^2+8)(4-x^2)^{\frac{3}{2}} + C
\end{aligned}
$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$

$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{x}{2}\right)^2} = \pm\frac{\sqrt{4-x^2}}{2}$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \Rightarrow \cos\theta > 0$

$\therefore \cos\theta = \frac{\sqrt{4-x^2}}{2}$

Note : $\sqrt{a^2-x^2}$ is well-defined only when $a^2-x^2 \geq 0$, that means $-a<x<a$.

Also we have $-1 \leq \sin\theta \leq 1$ when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$,

so $-a \leq a\sin\theta \leq a$, that is the reason why we let $x = a\sin\theta$.

Think : How about $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$?

e.g. $\int \frac{\sqrt{x^2-4}}{x^3}\,dx$

$\int \frac{\sqrt{x^2-4}}{x^3}\,dx$

Let $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$= \int \frac{\sqrt{4\tan^2\theta}}{8\sec^3\theta}\,2\sec\theta\tan\theta\,d\theta$

$= \frac{1}{2}\int \sin^2\theta\,d\theta$

$= \frac{1}{4}\int 1-\cos 2\theta\,d\theta$

$= -\frac{1}{8}\sin 2\theta + \frac{\theta}{4} + C$

Ex: ⋮

$= -\frac{\sqrt{x^2-4}}{2x^2} + \frac{1}{4}\cos^{-1}\frac{2}{x} + C$

Ex: Show that, for $a>0$, $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} \pm \frac{1}{2}a^2\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) + C$

$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2 \pm a^2}\right| + C$

Integration by Parts

Recall: Let $u(x)$ and $v(x)$ be differentiable functions.

Product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrate both sides with respect to x:

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

OR: $\int u\,dv = uv - \int v\,du$

Integration by Parts: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

e.g. $\int x^2 \ln x\,dx = \int (\ln x)\, x^2\,dx$

$$= \int (\ln x)\frac{d}{dx}\left(\frac{x^3}{3}\right)dx \qquad \left(\text{Now, } u = \ln x,\ v = \frac{x^3}{3}\right)$$

$$= \int \ln x \, d\frac{x^3}{3}$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\, d(\ln x)$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\,\frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

(Verify the answer by differentiation!)

e.g. $\int x e^x dx$

Note : $\frac{d}{dx} e^x = e^x$

$e^x dx = de^x$

$$\int x e^x dx$$
$$= \int x \, de^x$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + C$$
$$= e^x(x-1) + C$$

Now, $u = x$, $v = e^x$

Remark: Why don't we try the following?

$$\int x e^x dx$$
$$= \int e^x x dx$$
$$= \int e^x d\left(\frac{x^2}{2}\right)$$
$$\vdots$$

What happens?

e.g. $\int x^2 e^x dx$

$$= \int x^2 de^x$$
$$= x^2 e^x - \int e^x dx^2$$
$$= x^2 e^x - \underbrace{\int 2x e^x dx}$$

Ex: $\vdots$ Apply Integration by parts again!

Ans : $e^x(x^2 - 2x + 2) + C$

Question : How to make a guess of $u(x)$ and $v(x)$?

Integration by Parts : $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

e.g. $\int x^2 \ln x \, dx = \int (\ln x) \, x^2 \, dx$

$$= \int (\ln x) \frac{d}{dx}\left(\frac{x^3}{3}\right) dx$$

Realize the integrand as a product of parts and make a guess of $u(x)$ and $v(x)$ such that one part can be realized as a function $u(x)$, another part is $v'(x)$

e.g. $\int x\sin 3x\,dx$

$$\int x\sin 3x\,dx$$
$$= \int x\,d\left(-\tfrac{1}{3}\cos 3x\right)$$
$$= -\tfrac{1}{3}x\cos 3x - \int -\tfrac{1}{3}\cos 3x\,dx$$
$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + C$$

Integration of Logarithmic Functions :

$\int \ln x\,dx = ?$ for $x>0$

Using Integration by part :

$$\int \ln x\,dx \qquad u=\ln x \quad v=x$$
$$= x\ln x - \int x\,d\ln x$$
$$= x\ln x - \int x\cdot\tfrac{1}{x}\,dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + C$$

Ex: $\int \log_a x\,dx = ?$

Hints : $\log_a x = \dfrac{\ln x}{\ln a}$

$$\int \log_a x\,dx = \frac{1}{\ln a}\int \ln x\,dx$$
$$= \frac{1}{\ln a}(x\ln x - x + C)$$
$$= x\frac{\ln x}{\ln a} - \frac{x}{\ln a} + \frac{C}{\ln a}$$
$$= x\log_a x - \frac{x}{\ln a} + C' \qquad C' = \frac{C}{\ln a}$$ just a constant !

e.g. (Transformed into the original Integral)

$$\int e^x \cos x \, dx$$

$$\begin{aligned}\int e^x \cos x \, dx &= \int e^x \, d\sin x \\ &= e^x \sin x - \int \sin x \, de^x \\ &= e^x \sin x - \int e^x \sin x \, dx \\ &= e^x \sin x - \int e^x \, d(-\cos x) \\ &= e^x \sin x - \left(-e^x \cos x - \int -\cos x \, de^x\right) \\ &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x \, dx\right) \\ &= e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x \, dx}_{\text{back to itself !}}\end{aligned}$$

Be careful of +/- !

$$\therefore 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C'$$

Don't forget !

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C \qquad \left(C = \frac{1}{2} C'\right)$$

e.g. $\int \sin(\ln x) \, dx$

$$\begin{aligned}&\int \sin(\ln x) \, dx \\ &= x \sin(\ln x) - \int x \, d\sin(\ln x) \\ &= x \sin(\ln x) - \int \cos(\ln x) \, dx \\ &= x \sin(\ln x) - \left(x\cos(\ln x) - \int x \, d\cos(\ln x)\right) \\ &= x \sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x) \, dx\end{aligned}$$

$$\therefore \int \sin(\ln x) \, dx = \frac{1}{2} x \left[\sin(\ln x) + \cos(\ln x)\right] + C$$

Reduction Formulae

Idea: Obtain a formula to reduce the complexity of the integrand.

e.g. Let $I_n = \int x^n e^x \, dx$, where n is a nonnegative integer.
Prove that $I_n = x^n e^x - n I_{n-1}$, for $n \geq 1$.

$$
\begin{aligned}
I_n &= \int x^n e^x \, dx \\
&= \int x^n \, de^x \\
&= x^n e^x - \int e^x \, dx^n \\
&= x^n e^x - \int n e^x x^{n-1} \, dx \\
&= x^n e^x - n I_{n-1}
\end{aligned}
$$

Note: $I_0 = \int e^x \, dx = e^x + C$

We can apply this formula repeatly until we see I_0:

$$
\begin{aligned}
\int x^3 e^x \, dx = I_3 &= x^3 e^x - 3 I_2 \\
&= x^3 e^x - 3(x^2 e^x - 2 I_1) \\
&= x^3 e^x - 3(x^2 e^x - 2(x e^x - 1 \cdot I_0)) \\
&= x^3 e^x - 3x^2 e^x + 3 \cdot 2 x e^x - 3 \cdot 2 \cdot 1 \cdot I_0 \\
&= x^3 e^x - 3x^2 e^x + 3 \cdot 2 x e^x - 3 \cdot 2 \cdot 1 \cdot e^x + C \\
&= x^3 e^x - P_1^3 x^2 e^x + P_2^3 x e^x - P_3^3 e^x + C \\
&= \left[\sum_{r=0}^{3} (-1)^r P_r^3 x^{3-r} e^x \right] + C
\end{aligned}
$$

In general, $\int x^n e^x \, dx = \left[\sum_{r=0}^{n} (-1)^r P_r^n x^{n-r} e^x \right] + C$ for $n \geq 1$.

The formula $I_n = x^n e^x - n I_{n-1}$ is called a reduction formula.

e.g. Let $I_n = \int \tan^n x \, dx$, where n is a nonnegative integer.

Show that $I_n = \underbrace{\frac{1}{n-1}\tan^{n-1}x} - I_{n-2}$ for $n \geq 2$.

Why / How do we get this?

$\int \tan^{n-2}x \, d\tan x$

$$
\begin{aligned}
I_n &= \int \tan^n x \, dx \\
&= \int \tan^{n-2}x \tan^2 x \, dx \\
&= \int \tan^{n-2}x \, (\sec^2 x - 1) \, dx \\
&= \int \tan^{n-2}x \sec^2 x \, dx - \int \tan^{n-2}x \, dx \\
&= \int \tan^{n-2}x \, d\tan x - I_{n-2} \\
&= \frac{1}{n-1}\tan^{n-1}x - I_{n-2}
\end{aligned}
$$

As we can see, the index n is decreased by 2, so we have two cases:

Case 1: start from an even integer $n = 2m$

$$
\begin{aligned}
I_{2m} &= \frac{1}{2m-1}\tan^{2m-1}x + I_{2m-2} \\
&= \frac{1}{2m-1}\tan^{2m-1}x + \frac{1}{2m-3}\tan^{2m-3}x + I_{2m-4} \\
&\vdots \\
&= \frac{1}{2m-1}\tan^{2m-1}x + \frac{1}{2m-3}\tan^{2m-3}x + \cdots + \frac{1}{3}\tan^3 x + \tan x + I_0 \qquad \text{(end at } I_0\text{)} \\
&= \frac{1}{2m-1}\tan^{2m-1}x + \frac{1}{2m-3}\tan^{2m-3}x + \cdots + \frac{1}{3}\tan^3 x + \tan x + x + C \qquad \left(I_0 = \int dx = x + C\right)
\end{aligned}
$$

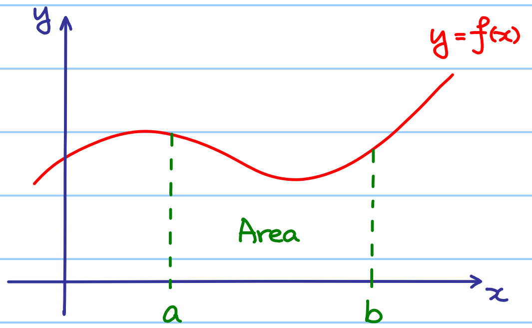
Case 2: start from an odd integer $n = 2m+1$

$$
\begin{aligned}
I_{2m+1} &= \frac{1}{2m}\tan^{2m}x + I_{2m-1} \\
&\vdots \\
&= \frac{1}{2m}\tan^{2m}x + \frac{1}{2m-2}\tan^{2m-2}x + \cdots + \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + I_1 \qquad \text{(end at } I_0\text{)} \\
&= \frac{1}{2m}\tan^{2m}x + \frac{1}{2m-2}\tan^{2m-2}x + \cdots + \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + \ln|\sec x| + C
\end{aligned}
$$

$\left(I_1 = \int \tan x \, dx = \ln|\sec x| + C\right)$

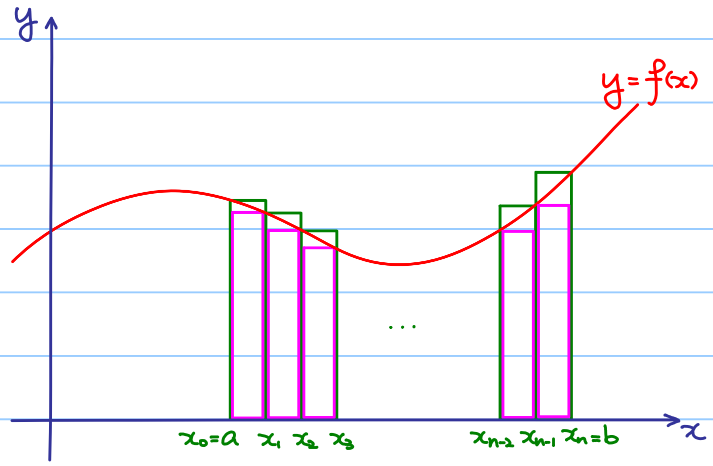
Definite Integration

Goal : Find the area of the region under the curve $y=f(x)$ over an interval $[a,b]$.



Riemann Sum

Area as the limit of a sum



Subdivide $[a,b]$ into n equal subintervals , $x_i - x_{i-1} = \Delta x$, $i = 1,2,3,\cdots,n$

$$\text{Upper sum} = \max_{x_0 \le \xi_1 \le x_1} f(\xi_1)\,\Delta x + \max_{x_1 \le \xi_2 \le x_2} f(\xi_2)\,\Delta x + \cdots + \max_{x_{n-1} \le \xi_n \le x_n} f(\xi_n)\,\Delta x$$

$$U_n = \sum_{i=1}^{n} \max_{x_{i-1} \le \xi_i \le x_i} f(\xi_i)\,\Delta x$$

$$\text{Lower sum} = \min_{x_0 \le \xi_1 \le x_1} f(\xi_1)\,\Delta x + \min_{x_1 \le \xi_2 \le x_2} f(\xi_2)\,\Delta x + \cdots + \min_{x_{n-1} \le \xi_n \le x_n} f(\xi_n)\,\Delta x$$

$$L_n = \sum_{i=1}^{n} \min_{x_{i-1} \le \xi_i \le x_i} f(\xi_i)\,\Delta x$$

Note : $L_n \le \text{Area} \le U_n$

Rough idea : $n \to \infty$, more rectangles , better approximation !

If $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = A$, we define the area to A. —— $(*)$

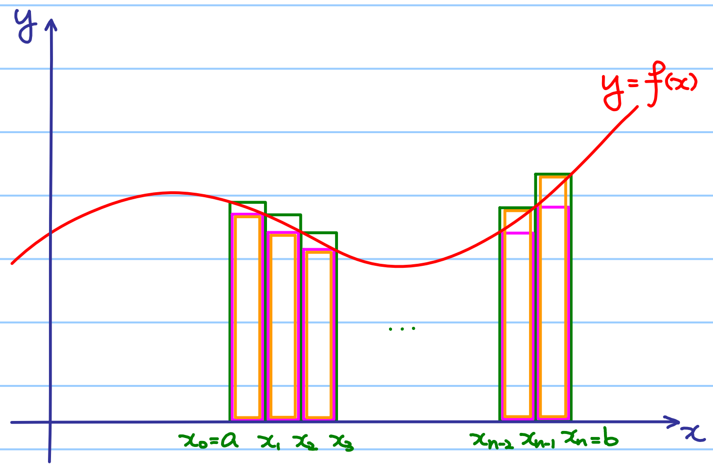
Remark :

1) If the area is defined, we denote it by $\int_a^b f(x)\,dx$.

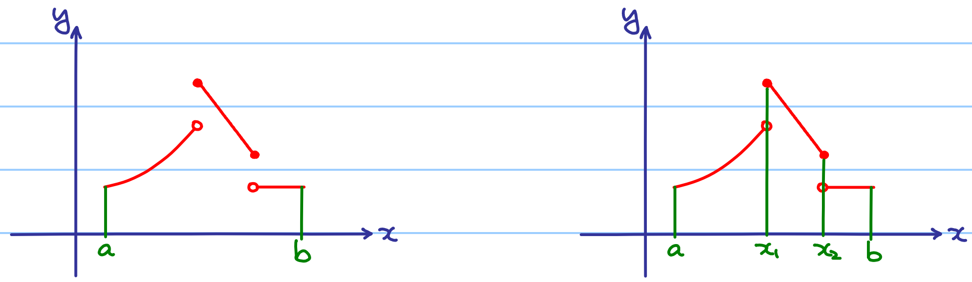
2) If $f(x)$ is a continuous function, $\int_a^b f(x)\,dx$ is well-defined for any $a \le b$.

3) Let $a_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(a+(b-a)\frac{i}{n}) \cdot \frac{b-a}{n}$, then $L_n \le a_n \le U_n$.

Now, we know $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = A$, so $\lim_{n\to\infty} a_n = A$

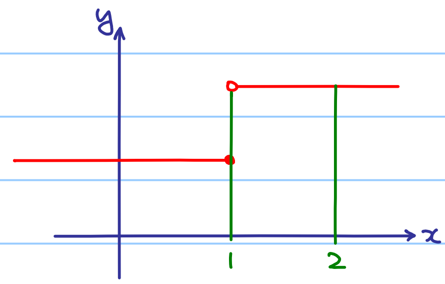


4) If f is a piecewise continuous on $[a,b]$, i.e. discontinuous only at finitely many points,
then $\int_a^b f(x)\,dx$ is defined as the following :



$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{b} f(x)\,dx$$

e.g. Let $f(x) = \begin{cases} 1 & \text{if} \quad x \le 1 \\ 2 & \text{if} \quad x > 1 \end{cases}$



$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= 1\times 1 + 2\times 1$$

$$= 3$$

Note : width of a point = 0

Rules for Definite Integrals :

Let $f(x)$, $g(x)$ be continuous (or piecewise continuous) functions.

Suppose $a \le b$.

1) If k is a constant, $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

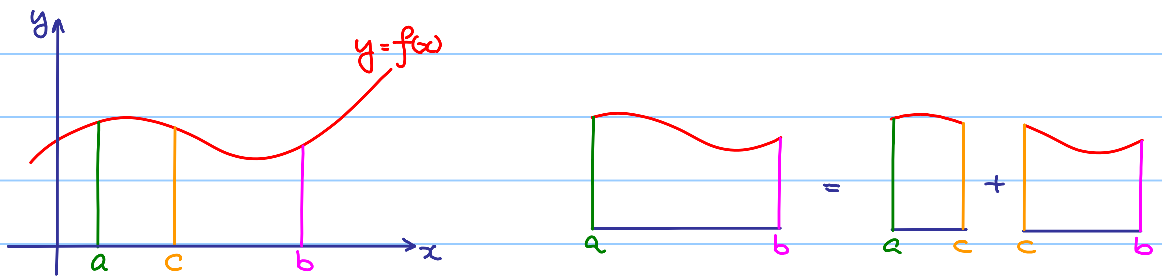
2) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3) $\int_a^a f(x)\,dx = 0$

4) $\int_b^a f(x)\,dx$ is defined to be $-\int_a^b f(x)\,dx$ (reverse direction)

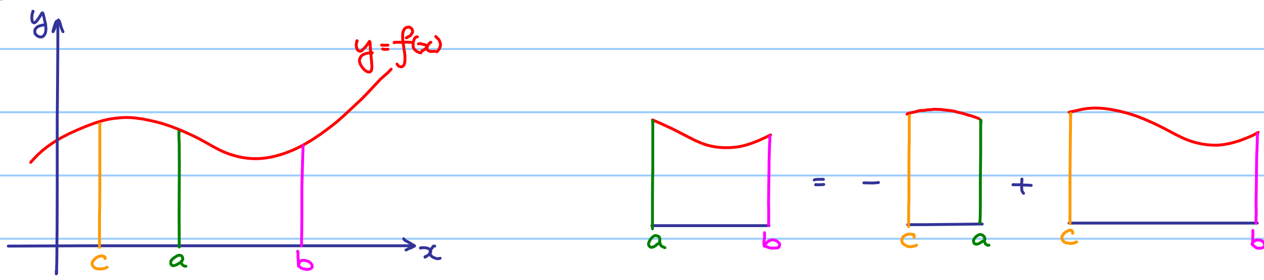
5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any c (subdivision)

If $a \le c \le b$,



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If $c < a \le b$,



$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{= -\int_c^a f(x)\,dx} + \int_c^b f(x)\,dx$$

Ex : Think why (5) is true if $a \le b < c$!

These properties are followed the definition (*).

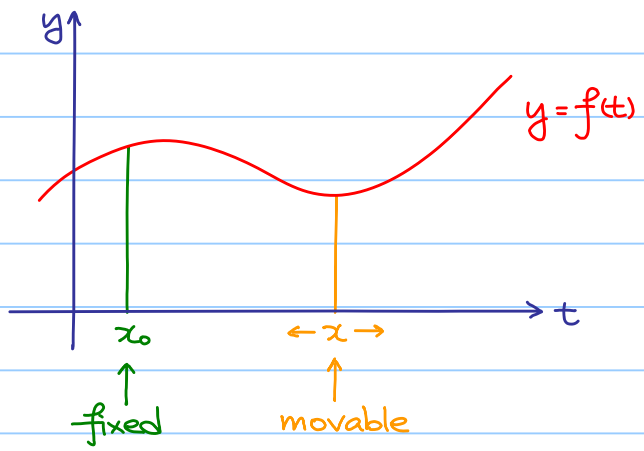
Computation of area :

NOT rely on the above limit , but Fundamental Theorem of Calculus !

Fundamental Theorem of Calculus :

Preparation :

Let $f(t)$ be a continuous function.



1) $\int_{x_0}^{x} f(t)\,dt$ is well defined for all $x \in \mathbb{R}$

2) What is a function? Roughly speaking, input x, output y.

Now, construct a new function $F(x)$ defined by

$F(x)$ = Area under the curve $y = f(t)$ over $[x_0, x]$

$= \int_{x_0}^{x} f(t)\,dt$

3) How about choosing another fixed point ?

Let $\tilde{F}(x) = \int_{x_1}^{x} f(t)\,dt$, what is the difference between $F(x)$ and $\tilde{F}(x)$?

In fact,

$$F(x) - \tilde{F}(x) = \int_{x_0}^{x} f(t)\,dt - \int_{x_1}^{x} f(t)\,dt$$

$$= \int_{x_0}^{x} f(t)\,dt + \int_{x}^{x_1} f(t)\,dt$$

$$= \int_{x_0}^{x_1} f(t)\,dt \quad \text{which is a constant .}$$

Fundamental Theorem of Calculus :

Let $f(t)$ be a continuous function, x_0 be a fixed point.

Suppose $F(x)$ is a function defined by

$$F(x) = \int_{x_0}^{x} f(t)\,dt,$$

then $F(x)$ is a differentiable function and $F'(x) = f(x)$.

(i.e. $F(x)$ is an antiderivative of $f(x)$.)

1) Direct consequence : $\int_a^b f(x)\,dx = \int_{x_0}^{b} f(x)\,dx - \int_{x_0}^{a} f(x)\,dx$

$$= F(b) - F(a)$$

i.e. if we know how to compute antiderivative of $f(x)$,

then we know how to find $\int_a^b f(t)\,dt$.

2) Wait ! Antiderivative of $f(x)$ is NOT unique, but unique up to a constant.

Which one should we pick ?

If $\tilde{F}(x) = \int_{x_1}^{x} f(t)\,dt$, then $\tilde{F}(x)$ is another antiderivative of $f(x)$.

In fact, it is NOT surprising, we know $F(x) - \tilde{F}(x)$ is a constant.

Also, $\int_a^b f(x)\,dx = \int_{x_1}^{b} f(x)\,dx - \int_{x_1}^{a} f(x)\,dx$

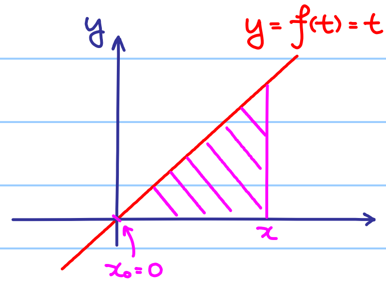
$$= \tilde{F}(b) - \tilde{F}(a)$$

Therefore, we can pick anyone !

e.g. (Verification of Fundamental Theorem of Calculus)

$f(t) = t$, $x_0 = 0$

$f(x) = x$



$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

$= \text{Area of the shaded triangle}$

$= \frac{1}{2}x^2$

Note : We have $F'(x) = f(x)$.

e.g. $f(x) = x+1$

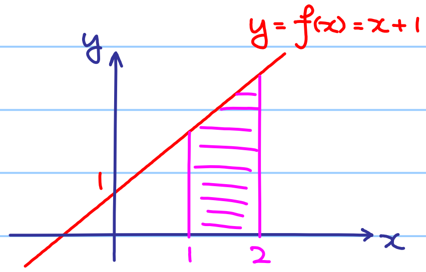
Antiderivative of $f(x) = \int x+1\,dx = \frac{x^2}{2} + x + C$

Choose $C = 0$, let $F(x) = \frac{x^2}{2} + x$

Area of the shaded region $= \int_1^2 f(x)\,dx = F(2) - F(1)$

$= 4 - \frac{3}{2}$

$= \frac{5}{2}$



What we write :

$\int_1^2 f(x)\,dx = \left[\frac{x^2}{2} + x\right]_1^2$

$= \underbrace{\left(\frac{2^2}{2} + 2\right)}_{F(2)} - \underbrace{\left(\frac{1^2}{2} + 1\right)}_{F(1)} = 4 - \frac{3}{2} = \frac{5}{2}$

e.g. $f(x) = x^2$

Area of the shaded region $= \int_0^1 f(x)\,dx$

$= \left[\frac{x^3}{3}\right]_0^1$

$= \left(\frac{1^3}{3}\right) - \left(\frac{0^3}{3}\right)$

$= \frac{1}{3}$