Integration of Trigonometric Functions :

Ex : Show that

a) $\int \sin px \, dx = -\frac{1}{p} \cos px + C$

b) $\int \cos px \, dx = \frac{1}{p} \sin px + C$

- $\int \sin px \cos qx \, dx$, $\int \sin px \sin qx \, dx$, $\int \cos px \cos qx \, dx$

Recall : $\sin px \cos qx = \frac{1}{2}[\sin(p+q)x + \sin(p-q)x]$

$\cos px \cos qx = \frac{1}{2}[\cos(p+q)x + \cos(p-q)x]$

$\sin px \sin qx = -\frac{1}{2}[\cos(p+q)x - \cos(p-q)x]$

We know how to integrate RHS !

e.g. $\int \sin 5x \cos 3x \, dx$

$= \frac{1}{2}\int \sin 8x + \sin 2x \, dx$

$= \frac{1}{2}\left(-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right) + C$

$= -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$

In particular, $\cos^2 px = \frac{1}{2}(1+\cos 2px)$

$\sin^2 px = \frac{1}{2}(1-\cos 2px)$

e.g. $\int \cos x \cos^2 3x \, dx$

$= \int \cos x \left[\frac{1}{2}(1+\cos 6x)\right] dx$

$= \frac{1}{2}\int \cos x \, dx + \frac{1}{2}\int \cos x \cos 6x \, dx$

$= \frac{1}{2}\int \cos x \, dx + \frac{1}{4}\int \cos 7x + \cos 5x \, dx$

$= \frac{\sin x}{2} + \frac{\sin 7x}{28} + \frac{\sin 5x}{10} + C$

Ex: Find $\int \sin x \sin 3x \sin 6x \, dx$

Ans: $\frac{\cos 10x}{40} + \frac{\cos 2x}{8} - \frac{\cos 8x}{10} - \frac{\cos 4x}{16} + C$

- $\int \sin^m x \cos^n x \, dx$

Case 1 : m is odd

Apply : $\sin x \, dx = -d\cos x$ and $\sin^2 x = 1 - \cos^2 x$

e.g. $\int \sin^3 x \cos^2 x \, dx$

$= \int \sin^2 x \sin x \cos^2 x \, dx$

$= -\int \sin^2 x \cos^2 x \, d\cos x$

$= -\int (1-\cos^2 x)\cos^2 x \, d\cos x$

$= \int -\cos^2 x + \cos^4 x \, d\cos x$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

Case 2: n is odd

Similar to case 1

Apply : $\cos x \, dx = d \sin x$ and $\cos^2 x = 1 - \sin^2 x$

Ex : $\int \sin^4 x \cos^3 x \, dx$

$= \int \sin^4 x \cos^2 x \cos x \, dx$

$= \int \sin^4 x (1 - \sin^2 x) \, d \sin x$

$\vdots$

$= \dfrac{\sin^5 x}{5} - \dfrac{\sin^7 x}{7} + C$

Case 3: m and n are even.

Apply: $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin x \cos x = \frac{1}{2}\sin 2x$

e.g. $\int \sin^2 x \cos^4 x \, dx$

$$= \int (\sin x \cos x)^2 \cos^2 x \, dx$$

$$= \int \left(\frac{1}{4}\sin^2 2x\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\underbrace{\int \sin^2 2x \, dx}_{\text{case 3 again}} + \frac{1}{8}\underbrace{\int \sin^2 2x \cos 2x \, dx}_{\text{reduce to case 1}}$$

$$= \frac{1}{16}\int 1-\cos 4x \, dx + \frac{1}{8}\int \sin^2 2x \, \frac{1}{2} \, d\sin 2x$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$$

- $\int \tan^m x \sec^n x \, dx$

Case 1 : m is odd

Apply : $\tan x \sec x \, dx = d \sec x$ and $\tan^2 x = 1 - \sec^2 x$

e.g. $\int \tan^3 x \sec^4 x \, dx$

$$= \int \tan^2 x \tan x \sec^3 x \sec x \, dx$$

$$= \int \tan^2 x \sec^3 x \, d\sec x$$

$$= \int (\sec^2 x - 1) \sec^3 x \, d\sec x$$

$$= \int \sec^5 x - \sec^3 x \, d\sec x$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

Case 2: n is even

Similar to case 1

Apply : $\sec^2 x \, dx = d\tan x$ and $\sec^2 x = 1 + \tan^2 x$

Ex: $\int \tan^4 x \sec^4 x \, dx$

$= \int \tan^4 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^4 x (1 + \tan^2 x) \, d\tan x$

$\vdots$

$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$

Case 3: m is even and n is odd

Using integration by parts, later!

- $\int \csc^m x \cot^n x \, dx$

Similarly, apply

$$\csc^2 x = -d\cot x$$

$$\csc x \cot x = -d\csc x$$

$$1+\cot^2 x = \csc^2 x$$

Ex: Find

a) $\int \csc^6 x \cot^4 x \, dx$ Ans: $-\frac{\cot^9 x}{9} - \frac{2\cot^7 x}{7} - \frac{\cot^5 x}{5} + C$

b) $\int \csc^5 x \cot^3 x \, dx$ $-\frac{\csc^7 x}{7} + \frac{\csc^5 x}{5} + C$