Week 9 (Day 1)

1510_h

Topics

- Language of Differentials
- By Parts
- Integration by Substitution/Trig. Sub./t-substitution (optional)
- Partial Fraction
- Fundamental Theorem of Calculus

  (The red-colored items have not been covered yet)

Introduction

The goal now is to find systematic methods to solve (*) $F'(x) = f(x)$ or equivalently, $F(x) = \int f(x)dx$ (**)

The equivalence of (*) and (**) comes from the following **Language of Differentials.**

As mentioned above, for any differentiable function $u(x)$ one can "formally" define

$$du(x) = u'(x)dx$$

Some Examples of Differentials

1)

$$dx^n = nx^{n-1}dx$$

2) $d\left(\frac{\sin(4x)}{x}\right) = \left(\frac{x\left(\frac{d\sin(4x)}{dx}\right)-\sin(4x)\left(\frac{dx}{dx}\right)}{x^2}\right)dx = \left(\frac{4x\cos 4x-\sin 4x}{x^2}\right)dx$

General Rules for Differentials (we omit the variable in the functions for simplicity)

$d(f+g) = df + dg$

$d(kf) = kdf$, k is a constant

$d(fg) = fdg + gdf$

$d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$

etc.

Using this "language of differentials" it's quite easy to write down some "methods" to compute indefinite integrals.

Method of Integration by Parts

Some of you might have notice that we haven't discussed the rule for finding the "indefinite integrals of (product) of two functions". Why? The reason is because it is quite involved.

The rule is known as "Integration by parts" which is (in most concise writing):

$$\int f dg = fg - \int g df$$

or $\int f(x)dg(x) = f(x)g(x) - \int g(x)df(x)$, or $\int f(x)dg(x) + g(x)df(x) = f(x)g(x)$, or $\int f(x)g'(x)dx + g(x)f'(x)dx = f(x)g(x)$.

Examples on how to apply Integration by parts

1) Find $\int xe^x dx$.

The idea is to "read" this as $\int f dg$. Which function can we choose for $f(x)$, for $dg(x)$?

Many choices: (Choice 1) $f(x) = x$, $g'(x)dx = e^x dx$ (or $dg(x) = de^x$).

This choice gives $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

Question: Try (Choice 2) $f(x) = e^x$, $g'(x)dx = xdx$. What would this choice lead to?

2) Find $I = \int e^x \sin x \, dx$.

Choices: (1) $f(x) = e^x, g'(x)dx = \sin x \, dx$. This gives $g(x) = -\cos x$. Hence we have $I = fg - \int gdf = -e^x \cos x - \int (-\cos x) \cdot e^x dx = -e^x \cos x + \int \cos x \, e^x dx = -e^x \cos x + \int e^x d \sin x$

$$= -e^x \cos x + (e^x \sin x - \int \sin x \, e^x dx)$$

$$I = -e^x \cos x + e^x \sin x - I$$

Hence $I = \frac{1}{2}(-e^x \cos x + e^x \sin x) + C$.

3) Reduction Formula – this example is about "reduction formula".

(A reduction formula is a formula relating a more complicated integral to a less complicated one)

Example: Find $I_n = \int x^n e^x dx$.

Solution: $I_n = \int x^n de^x$ by choosing $f(x) = x^n$ and $dg(x) = de^x$

Using the formula $\int f dg = -\int gdf + fg$ we obtain $I_n = \int x^n de^x$
$= -\int e^x dx^n + x^n e^x = -\int nx^{n-1} e^x dx + x^n e^x = -nI_{n-1} + x^n e^x$

which relates I_n to I_{n-1}.

Note that when $n=1$, this is just our first example above.

The Paradox $\mathbf{0=1}$ via Integration by Parts

In the following, we will show that $0=1$ by using integration by parts. This example shows that the constant is important.

Example

Let's compute $\int \frac{1}{x}\,dx$ by using integration by parts. To do this, we choose $f(x)=\frac{1}{x}$

and $dg(x)=1\cdot dx$. Then we have $\int f dg = \int \frac{1}{x}dx = -\int g df + fg$

$= -\int x d(x^{-1}) + x(x^{-1}) = -\int x(-x^{-2})dx + 1 = \int \frac{1}{x}\,dx + 1$

Cancelling the terms $\int \frac{1}{x}dx$, we get $0=1$.

Remark:

One explanation of this paradox is that the constant of indefinite integral is important. If we choose a suitable constant, the paradox can be resolved.

We will see more explanations later.

Explanation of the formula $\int \boldsymbol{f dg} = -\int \boldsymbol{g df} + \boldsymbol{fg}$

First note that the formula is equivalent to $\int (f dg + g df) = fg$ (1)

$$\Leftrightarrow \int d(fg) = fg$$

To prove (1), we start from the product rule for differentiation, i.e.

$$fg' + f'g = (fg)'$$

$$\Leftrightarrow f\frac{dg}{dx} + g\frac{df}{dx} = \frac{d(fg)}{dx}$$

$$\Leftrightarrow f\frac{dg}{dx}dx + g\frac{df}{dx}dx = \frac{d(fg)}{dx}dx$$

$$\Leftrightarrow \int f\frac{dg}{dx}dx + \int g\frac{df}{dx}dx = \int \frac{d(fg)}{dx}dx \qquad (***)$$

But now $f\frac{dg}{dx}dx = fdg,\ \ g\frac{df}{dx}dx = gdf, \frac{d(fg)}{dx}dx = d(fg),$ so (***) gives

$$\int fdg + gdf = fg$$

which is just (1) above, as was required to be proved.

Other Methods of Computing $\int f(x)dx$

Substitution Method (Case 1. Simple Substitution)

$$\int (1+x)^{100}dx$$

A straight-forward but tedious method to compute this indefinite integral is to expand $(1+x)^{100}$ and the compute them term by term.

A much better method is to let $u = 1 + x$ to get $du = (1+x)'dx = dx$ and hence

$$\int (1+x)^{100}dx = \int u^{100}du = \frac{u^{101}}{101} + C$$

Another Example

Find $\int \frac{dx}{x\ln x},\quad x > 1$

Solution: The idea is to "see" that $\frac{dx}{x} = d\ln x$. Hence we can let $u = \ln x$ which

gives $\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln u + C = \ln(\ln x) + C.$

Some Theory

The reason why the above method works is due to the Chain Rule (which we will not elaborate here).

In general, in using the "substitution method", the (indefinite) integral takes the form

$$\int f(g(x))g'(x)dx$$

so that we can rewrite the term $g'(x)dx$ as du (after letting $u = g(x)$). Doing this, the integral $\int f(g(x))g'(x)dx$ becomes $\int f(u)du$, which may be easier to compute.

Examples (for $\int f(g(x))g'(x)dx = \int f(u)du$)

1) $\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}\frac{1}{x}dx = \int \frac{1}{\ln x}d(\ln x) = \int f(u)du$, where now $f(u) = \frac{1}{\ln x}$ and $u = \ln x$.
2) $\int (1+x^2)^7 xdx$

   Now the term $xdx = d\left(\frac{x^2}{2}\right) = \left(\frac{1}{2}\right)dx^2$. We can now let $u = x^2$ and obtain

$$\int (1+x^2)^7 xdx = \int (1+u)^7\left(\frac{1}{2}\right)du = \cdots$$

Important Remark:

In the above paragraph, we see that the function to be integrated is now

$$f(g(x))g'(x)$$

In order that we can get $\int f(g(x))g'(x)dx$ (or equivalently, solve $F'(x) = f(g(x))g'(x)$), this function $f(g(x))g'(x)$ has to be continuous.

Substitution Method (Case 2) Trigonometric Substitution Method

A very important class of substitution is the "trig. sub.". They are there mainly to deal with integrals involving (i) a square root sign, (ii) a quadratic term inside the square root.

Example

Find $\int \frac{dx}{\sqrt{a^2-x^2}}$, $a > 0$

Main Idea: Completing square, i.e. to rewrite the expression inside the square root sign to become "something squared". There are many ways to get this, one way is to let $x = a\sin t$.

Doing this, we obtain $x = a\sin t \quad gives \quad dx = a\cos t\, dt$

Also, we have $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2 t} = a|\cos t|$

Therefore $\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{a|\cos t|}$

And $\frac{dx}{\sqrt{a^2-x^2}} = \frac{a\cos t\, dt}{a|\cos t|} =?$

Supposing the case "$\cos t > 0$" is true, then we have

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int dt = t + C = \arcsin\left(\frac{x}{a}\right) + C$$

Remark:

This question already reveals to us that "blind" use of "formal" computation is not enough, one sometimes have to think about the domain of integration. (A correct choice of "domain of integration (More later)" will lead to $|\cos t| = \cos t$

Another Example

Find $\int \frac{1}{1+x+x^2} dx$.

First we have to rewrite it in the form $\int \frac{du}{a^2+b^2u^2}$ for some suitable choices of a and b.

How to do it?

Completing square again. $1 + x + x^2 = \left(\frac{3}{4}\right) + \left(x + \left(\frac{1}{2}\right)\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(x + \left(\frac{1}{2}\right)\right)^2$

Now we can let $u = x + \left(\frac{1}{2}\right)$ and get $\frac{1}{1+x+x^2} dx = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2 + u^2} du.$

Next, we perform another completing square to make the denominator a "complete square".

To do this, we let $u = \frac{\sqrt{3}}{2}\tan\theta$, which leads to $du = \frac{\sqrt{3}}{2}\sec^2\theta \, d\theta.$

On the other hand, $\left(\frac{\sqrt{3}}{2}\right)^2 + u^2 = \left(\frac{\sqrt{3}}{2}\right)^2 (1 + \tan^2\theta) = \left(\frac{\sqrt{3}}{2}\right)^2 \sec^2\theta$

Putting everything together, we have $\frac{d\theta}{\frac{\sqrt{3}}{2}}$

Hence $\int \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2 + u^2} du = \int \frac{d\theta}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}\theta + C = \frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}u + C$

$$= \frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right) + C$$