MATH 2550

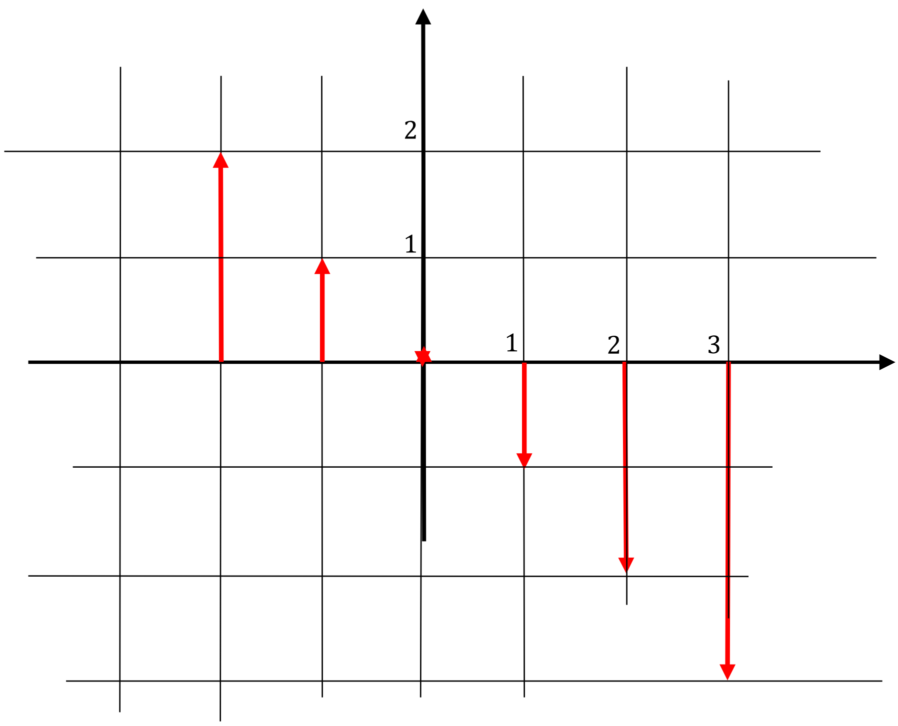
Notes 2

(Keywords: line integral of a vector field, Green's Theorem for a rectangle, Green's Theorem in general, orientation of a curve.)

- (Line Integral over a curve C of a vector field $\vec{F}$) Before everything, let's define precisely the "nouns" involved:
  - A vector field $\vec{F}$ in the plane (i.e. $\mathbb{R}^2$) is an (association of a "displacement vector" $\vec{F}$ to each given point $(x, y)^t$ in the plane).
  - Now the above sentence means something depicted by the following diagram:

$$(x, y)^t \mapsto \vec{F}(x, y)$$

  - Example: Consider this one, $\vec{F}(x, y) = y\hat{\imath} - x\hat{\jmath}$, what does this mean? It means: "To each given point $(x, y)^t$ (position vector!) in the plane we attach to each (as "tail") the (displacement vector) $(y, -x)^t$."



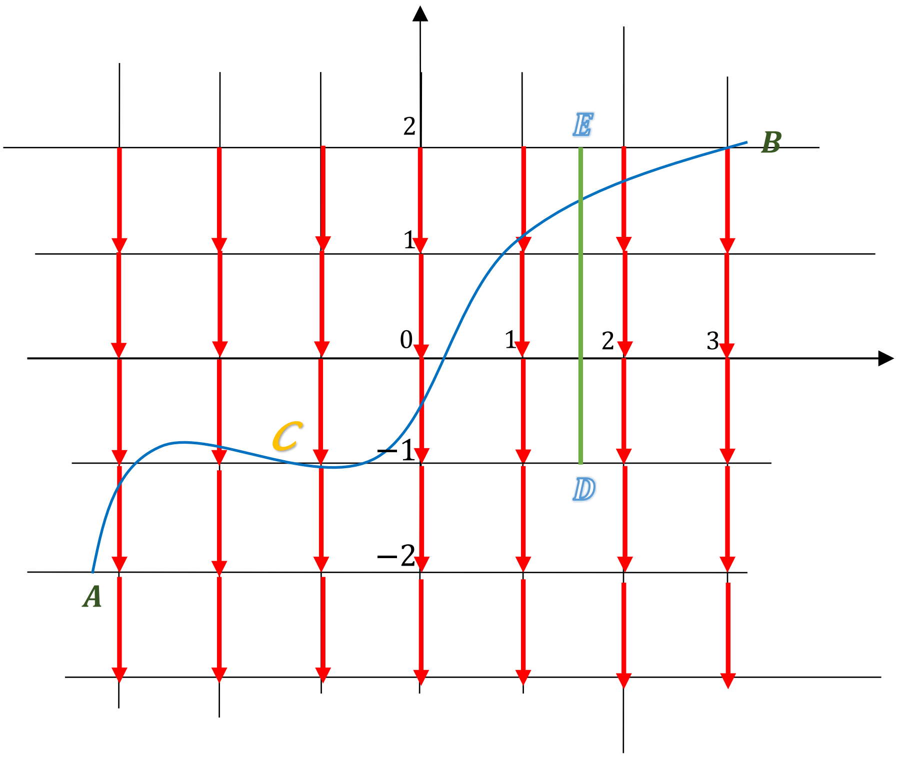
The above diagram illustrate (partially) how the vector field $\vec{F}$ looks like.

- (Quick Exercise) Draw on a sheet of graph paper the following (planar) vector field ("planar vector field" = vector field in the plane):

$$\vec{F}(x, y) = \left(\frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}}\right)^t$$

- (Remark) The following website helps you to easily "visualize" any planar vector field (however without "arrows"!) You should try it out yourself: https://www.geogebra.org/m/utcMvuUy .
  - To see our example 1 above, just (i) input y in the field to the right-hand side of $x' = u(x,y)\cdots$,
  - Similarly, (ii) input $-x$ in the field to the right-hand side of $y' = v(x,y)\cdots$

- (line integral of $\vec{F}$ over the curve C) To understand this phrase, we turn to Physics. In Physics, we have the notion of "work done" under a "force field". A simple example of this is



In this diagram, we have a "constant magnitude" (downward pointing) gravitational field given by $\vec{F}(x,y) = -mg\,\hat{\jmath}$. Suppose we want to move a particle (against) this vector field from the point A to the point B along the C.
The question is this:
"What is the work done in order to achieve this?"

- (Remark) A similar question is: "Find the (potential energy) needed to overcome the gravitational force if one moves a particle of mass m from D to E ."
(Answer) It is $mg\big(2-(-1)\big) = 3mg$. Another way of saying this is:

  It is the quantity $-\int_C \vec{F}\cdot d\vec{r}$ where the curve is the "green" line joining D to E and $\vec{F} = -mg\hat{\jmath}$.

- Green's Theorem. This is one of the most important theorem in Vector Analysis. The theorem goes like this.

  Let R be a region in the plane, C its boundary (oriented "counterclockwise"), then the following formula holds $\oint_C Adx + Bdy = \iint_R (B_x - A_y) dxdy$.

- (Remarks) (1) To understand this formula, one has to understand (i) the right-hand side. The right-hand side is a double integral.
  (2) To prove this formula, one can (i) prove the formula for the special case when R is a rectangle, then
  (3) insert lots of rectangles inside R, numbered by the index i. This way, we obtain lots of formulas of the form $\oint_{C_i} Adx + Bdy = \iint_{R_i} (B_x - A_y) dxdy$.

  (4) It is important to note that (i) $\int_{C_i} Adx + Bdy = \int_{C_i^-} Adx + Bdy$, if by C_i^- we mean the "curve C_i traveling in the reverse direction".
  (5) Using (4), we obtain $\int_{C_i} Adx + Bdy + \int_{C_i^-} Adx + Bdy = 0$. This means if "travel along C_i in one direction and then come back along the same curve in reverse direction, the resulting integral is zero".
  (6) It is worth remembering that a region may have more than one boundary curve. For example, the region $R = \{(x, y)^t : 1 \le x^2 + y^2 \le 4\}$ has two boundary curves, namely the circle with radius 1 and with radius 2, both centered at the origin.
- After summing these formulas together, we get a good approximation of any "nice" region R. In conclusion, we can prove the Green's Theorem which says: $\oint_C Adx + Bdy = \iint_R (B_x - A_y) dxdy$.
- (Remarks) 1. We have never made precise the word "nice". It can actually be made mathematically very precise.
  2. If you look at the Green's Theorem as written above, it may occur to you that the right-hand side is somewhat asymmetric, because (i) the 1st term is B_x and (ii) the 2nd term has a "minus" sign before it.

- (Divergence Theorem in the plane) A way to answer (ii) in the remark above is to "do a rotation" in the Green's Theorem (GT in short). This will be discussed in the next lecture notes.