MATH2060 Solution 1

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6.1 Q4

$\forall\varepsilon > 0$, take $\delta = \varepsilon$, for $0 < |x| < \delta$, if $x \in \mathbb{Q}$, then

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = \left|\frac{x^2 - 0}{x - 0} - 0\right| = |x| < \varepsilon;$$

if $x \notin \mathbb{Q}$, then

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = \left|\frac{0 - 0}{x - 0} - 0\right| = 0 < \varepsilon$$

Therefore, f is differentiable at $x = 0$, and $f'(0) = 0$.

6.1 Q10

Let $g_1(x) = x^2$, $f(x) = \frac{1}{x^2}$, $h(x) = \sin x$, then $g(x) = g_1(x) \cdot (h \circ f)(x)$.

First, for $x \neq 0$, f is differentiable at x, and h is differentiable at $f(x)$. By the chain rule, $h \circ f$ is differentiable at x. Since g_1 and $h \circ f$ are both differentiable at x, then by the product rule, $g_1 \cdot (h \circ f)$ is differentiable at x. And

$$\begin{aligned}
g'(x) &= g_1'(x) \cdot (h \circ f)(x) + g_1(x) \cdot (h \circ f)'(x) \\
&= g_1'(x) \cdot (h \circ f)(x) + g_1(x) h'(f(x)) f'(x) \\
&= 2x \sin \frac{1}{x^2} + x^2 \cos \frac{1}{x^2} \cdot (-\frac{2}{x^3}) \\
&= 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}
\end{aligned}$$

Second, for $x = 0$, $\forall\varepsilon > 0$, take $\delta = \varepsilon$, then for $0 < |x| < \delta$, we have

$$\left|\frac{g(x) - g(0)}{x - 0} - 0\right| = \left|\frac{x^2 \sin \frac{1}{x^2} - 0}{x - 0}\right| = \left|x \sin \frac{1}{x^2}\right| \leq |x| < \varepsilon$$

This shows $\lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = 0$. Take a sequence $\{x_n\}_n$ with $x_n = \frac{1}{\sqrt{2\pi n}}$, $n \in \mathbb{N}$. Clearly, $x_n \in [-1, 1]$, $\forall n \in \mathbb{N}$. Then

$$g'(x_n) = \frac{2}{\sqrt{2\pi n}} \sin 2\pi n - 2\sqrt{2\pi n} \cos 2\pi n = -2\sqrt{2\pi n}.$$

$\forall M > 0$, set $N = [\frac{M^2}{8\pi}] + 1$, then $\forall n \geq N$, $|g'(x_n)| = |-2\sqrt{2\pi n}| \geq M$. It shows g' is not bounded on the interval $[-1, 1]$.

6.1 Q15

Denote the function f and g repectively by $f = \cos : [0, \pi] \to [-1, 1]$ and $g = \arccos : [-1, 1] \to [0, \pi]$. For $x \in (0, \pi)$, let $y = \cos x$, since $f(x)$ is differentiable at x, and $f'(x) = \sin(x) \neq 0$, then by Theorem 6.1.8 in the textbook, we have

$$\arccos y = \frac{1}{(\cos x)'} = -\frac{1}{\sin x} = -\frac{1}{\sqrt{1 - \cos^2 x}} = -\frac{1}{\sqrt{1 - y^2}}$$

where $y \in (-1, 1)$. For $y = 1$, then $g(1) = \arccos 1 = 0$, and $f'(0) = \cos'(0) = 0$. Suppose $g = \arccos$ is differentiable at $y = 1$. Since $g \circ f(x) = x$, $x \in [0, \pi)$, then by the chain rule, we get $g'(f(0))f'(0) = 1$. However, $f'(0) = 0$, which leads to a contradiction.

Therefore, $g = \arccos$ is not differentiable at $y = 1$. For the case $y = -1$, we apply the similar argument.

6.2 Q9

For $x \neq 0$, note that $\sin \frac{1}{x} \geq -1$, then

$$f(x) = 2x^4 + x^4 \sin \frac{1}{x} \geq 2x^4 - x^4 = x^4 \geq 0$$

Since $f(0) = 0$, then f has an absolutely minimum at $x = 0$. For $x \neq 0$, we can get

$$\begin{aligned} f'(x) &= 8x^3 + 4x^3 \sin \frac{1}{x} - x^2 \cos \frac{1}{x} \\ &= x^2 \left(8x + 4x \sin \frac{1}{x} - \cos \frac{1}{x} \right) \end{aligned}$$

Take a sequence $\{x_n\}_{n \in \mathbb{N}}$ with $x_n = \frac{1}{\pi n}$, $n \in \mathbb{N}$. Then

$$\begin{aligned} f'(x_n) &= \left(\frac{1}{\pi n} \right)^2 \left(\frac{8}{\pi n} + \frac{4}{\pi n} \sin \pi n - \cos \pi n \right) \\ &= \left(\frac{1}{\pi n} \right)^2 \left(\frac{8}{\pi n} - (-1)^n \right) \end{aligned}$$

For any neighbourhood V of 0, without loss of generality, we consider the case $V = (-\delta, \delta)$, for some $\delta > 0$. We set $N = \max\{[\frac{1}{\pi\delta}] + 1, [\frac{8}{\pi}] + 1\}$, then $\forall n \geq N$,

$x_n = \frac{1}{\pi n} \in V = (-\delta, \delta)$. And if we take some n_1, such that $n_1 \geq N$ and let n_1 be even, then

$$f'(x_{n_1}) = \left(\frac{1}{\pi n_1}\right)^2 \left(\frac{8}{\pi n_1} - (-1)^{n_1}\right) = \left(\frac{1}{\pi n_1}\right)^2 \left(\frac{8}{\pi n_1} - 1\right) < 0$$

On the other hand, if we take some n_2, such that $n_2 \geq N$ and let n_2 be odd, then

$$f'(x_{n_2}) = \left(\frac{1}{\pi n_2}\right)^2 \left(\frac{8}{\pi n_2} - (-1)^{n_2}\right) = \left(\frac{1}{\pi n_2}\right)^2 \left(\frac{8}{\pi n_2} + 1\right) > 0$$

Therefore, f' has both positive and negative values in V.

6.2 Q10

$\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{2}$, then for $0 < |x| < \delta$,

$$\left|\frac{g(x) - g(0)}{x - 0} - 1\right| = \left|\frac{x + 2x^2 \sin\frac{1}{x} - 0}{x - 0} - 1\right| = \left|2x \sin\frac{1}{x}\right| \leq |2x| < \varepsilon$$

It shows $g'(0) = 1$. For $x \neq 0$, we have

$$\begin{aligned} g'(x) &= 1 + 4x \sin\frac{1}{x} + 2x^2 \cos\frac{1}{x}(-\frac{1}{x^2}) \\ &= 1 + 4x \sin\frac{1}{x} - 2\cos\frac{1}{x} \end{aligned}$$

Take a sequence $\{x_n\}_{n \in \mathbb{N}}$ with $x_n = \frac{1}{\pi n}$, $n \in \mathbb{N}$. Then

$$g'(x_n) = 1 + \frac{4}{\pi n} \sin \pi n - 2\cos \pi n = 1 - 2 \cdot (-1)^n$$

For any neighbourhood V of the point 0, without loss of generality, we consider the case $V = (-\delta, \delta)$, for some $\delta > 0$. Set $N = [\frac{1}{\pi\delta}] + 1$, then $\forall n \geq N$, $x_n \in V = (-\delta, \delta)$. If $n \geq N$, and n is even, then $g'(x_n) = 1 - 2 = -1 < 0$; if $n \geq N$, and n is odd, then $g'(x_n) = 1 - 2 \cdot (-1) = 3 > 0$. Therefore, g' takes on both positive and negative values in V.

6.2 Q13

Take a, $b \in I$, and $a < b$. Since I is an interval, then $[a, b] \subset I$. Clearly, f is differentiable on (a, b) and f is continuous on $[a, b]$. By Mean Value Theorem, there exists $c \in (a, b)$, such that

$$f(b) - f(a) = f'(c)(b - a)$$

Since $c \in I$, then $f'(c) > 0$, implying that $f(b) - f(a) > 0$. Then f is strictly increasing on I.