Solutions of Midterm Exam

1. (a) Since $4^2 - 4 \times 1 \times 1 > 0$, it is of the hyperbolic type.

   (b) Since $6^2 - 4 \times 9 \times 1 = 0$, it is of the parabolic type.

   (c) Since $12^2 - 4 \times 4 \times 9 = 0$, it is of the parabolic type.

2. (a) Fix (x, y). Let $z(s) = u(x+s, y+2s)$, then

$$z'(s) - 4z(s) = \mathrm{e}^{x+y+3s}.$$

   Since $z(-\frac{y}{2}) = \sin[(x-\frac{y}{2})^2]$,

$$\begin{aligned} u(x,y) = z(0) &= \sin[(x-\tfrac{y}{2})^2]\mathrm{e}^{2y} + \int_{-y/2}^{0} \mathrm{e}^{-4s}\mathrm{e}^{x+y+3s}\,\mathrm{d}s \\ &= \sin[(x-\tfrac{y}{2})^2]\mathrm{e}^{2y} + \mathrm{e}^{x+3y/2} - \mathrm{e}^{x+y}. \end{aligned}$$

   And it is easy to verify that $\sin[(x-\frac{y}{2})^2]\mathrm{e}^{2y} + \mathrm{e}^{x+3y/2} - \mathrm{e}^{x+y}$ is a solution.

   (b) Fix (t, x). Let $z(s) = u(t+s, x+\frac{3}{2}s)$, then

$$z'(s) = 0.$$

   Since $z(-t) = u(0, x-\frac{3}{2}t) = \sin(x-\frac{3}{2}t)$,

$$u(t,x) = z(0) = \sin(x - \tfrac{3}{2}t).$$

   And it is easy to verify that $\sin(x-\frac{3}{2}t)$ is a solution.

   (c) Fix (t, x). Solve

$$\begin{cases} t'(s) = x(s); \\ x'(s) = -t(s); \\ t(0) = t, \quad x(0) = x. \end{cases}$$

   Then we obtain $(t(s), x(s)) = (t\cos s + x\sin s, x\cos s - t\sin s)$. Let $z(s) = u(t(s), x(s))$, then

$$z'(s) = z(s).$$

   Since $z(-\arctan\frac{t}{x}) = x^2 + t^2$,

$$u(t,x) = z(0) = \mathrm{e}^{\arctan(t/x)}(x^2+t^2).$$

And it is easy to verify that $\mathrm{e}^{\arctan(t/x)}(x^2+t^2)$ is a solution.

3. (a) A well-posed problem should have the following three properties:

   - Existence: the problem has a solution;
   - Uniqueness: there is at most one solution;
   - Stability: solution depends continuously on the data given in the problem.

   (b) Since every constant is a solution, the problem does not have uniqueness. So it is not well-posed.

   (c) It is clear that u_n satisfies the initial and boundary conditions. $\partial_t u_n = -n\sin nx\mathrm{e}^{-n^2t}$ and $\partial_x u_n = -n\sin nx\mathrm{e}^{-n^2t}$. So $\partial_t u_n = \partial_x^2 u_n$ holds. Therefore u_n is a solutions to the problem.

$$E(t,u_n) = \frac{\pi}{2n^2}\mathrm{e}^{-2n^2t}.$$

   So $E(t,u_n)$ decreasingly tends to 0 as $t\to+\infty$ and increasingly tends to $+\infty$ as $t\to-\infty$.

   (d) By (c), $\frac{1}{n}\sin nx \to 0$ as $n\to\infty$, but $\sup_{x\in[0,\pi]}\frac{1}{n}|\sin nx|\mathrm{e}^{-n^2t} = \frac{1}{n}\mathrm{e}^{-n^2t}\to\infty$ as $n\to\infty$. So the problem does not have stability and is not well-posed.

4. (a) Since a solution to $v_t = v$ is e^t, we may consider $w = \mathrm{e}^{-t}v$. By substituting $v = \mathrm{e}^t w$ into the equation, we have

$$\mathrm{e}^t(w_t + w) = \mathrm{e}^t w_{xx} + \mathrm{e}^t w.$$

   So

$$w_t - w_{xx} = 0.$$

   Moreover, $w(0,x) = v(0,x) = \phi(x)$. Therefore,

$$w(t,x) = \int_{\mathbb{R}} S(t,x-y)\phi(y)\,\mathrm{d}y$$

   and

$$v(t,x) = \mathrm{e}^t\int_{\mathbb{R}} S(t,x-y)\phi(y)\,\mathrm{d}y.$$

   And it is easy to verify that the above v is a solution.

   (b) Let

$$\widetilde{\phi}(x) = \begin{cases}\phi(x) & x\geq 0;\\ -\phi(-x) & x<0.\end{cases}$$

Consider the solution to the initial data problem with initial data $\widetilde{\phi}$:

$$\begin{aligned} v(t,x) &= \int_{\mathbb{R}} S(t,x-y)\widetilde{\phi}(y)\,\mathrm{d}y \\ &= \int_0^\infty (S(t,x-y) - S(t,x+y))\phi(y)\,\mathrm{d}y. \end{aligned}$$

Then it is easy to verify that the above v is a solution.

(c) Let

$$\widetilde{\phi}(x) = \begin{cases} \phi(x) & x \geq 0; \\ \phi(-x) & x < 0. \end{cases}$$

Consider the solution the initial data problem with initial data $\widetilde{\phi}$:

$$\begin{aligned} v(t,x) &= \int_{\mathbb{R}} S(t,x-y)\widetilde{\phi}(y)\,\mathrm{d}y \\ &= \int_0^\infty (S(t,x-y) + S(t,x+y))\phi(y)\,\mathrm{d}y. \end{aligned}$$

Then it is easy to verify that the above v is a solution.

5. Let v solve

$$\begin{cases} \Delta v = 0 & \text{in } B_{1/2}; \\ v = u & \text{on } \partial B_{1/2}. \end{cases}$$

We claim that $u = v$ in $B_{1/2} \setminus \{0\}$. Indeed, we can consider $w = v - u$ in $B_{1/2} \setminus \{0\}$ and $M_r = \max_{\partial B_r} |w|$, where $0 < r < 1/2$. We observe that

$$|w(x)| \leq M_r \frac{\log|x|^{-1}}{\log r^{-1}} \quad \text{on } \partial B_r.$$

Note that w and $\log|x|^{-1}$ are harmonic in $B_{1/2} \setminus \overline{B_r}$. Hence the maximum principle implies

$$|w(x)| \leq M_r \frac{\log|x|^{-1}}{\log r^{-1}} \quad \text{for any } x \in B_{1/2} \setminus B_r.$$

Note also that, for all $r \in (0, 1/2)$,

$$M_r \leq \max_{\partial B_r} |u(x)| + \max_{\partial B_r} |v(x)| \leq \max_{\partial B_r} |u(x)| + \max_{\partial B_{1/2}} |u(x)|.$$

Combining the above estimates, we have for each fixed $x \neq 0$,

$$|w(x)| \leq \frac{\log|x|^{-1}}{\log r^{-1}} \left(\max_{\partial B_r} |u(x)| + \max_{\partial B_{1/2}} |u(x)| \right) \to 0 \text{ as } r \to 0,$$

that is $w = 0$ in $B_{1/2} \setminus \{0\}$. Therefore, u can be defined at 0 via v to make it be C^2 and harmonic in B_1.

6. Obviously, 0 is a solution. We only need to show that solution satisfying the conditions

is 0. First, by the mean value property, the condition

$$\lim_{r\to\infty}\left(\max_{|x|=r}\int_{B_1(x)} u(\xi)\,\mathrm{d}\xi\right)=0$$

is equivalent to

$$\lim_{r\to\infty}\max_{|x|=r} u(x)=0.$$

To use the Harnack inequality, by considering $-u$, we could assume that

$$\lim_{r\to\infty}\min_{|x|=r} u(x)=0.$$

Next we show that $u\geq 0$. Fix x. For arbitrary $\varepsilon>0$, suppose that

$$\min_{|x|=R} u>-\varepsilon,$$

where $R>|x|$. Then by applying the minimum principle to $\{1<|x|<R\}$,

$$u(x)\geq-\varepsilon.$$

Let $\varepsilon\to 0$, then we obtain $u(x)\geq 0$.

Next we prove a lemma that is a variant of the Harnack inequality.

Lemma 0.1 *Suppose that u is C^2, harmonic, and non-negative in $\{|x|>1\}$. Then there is a universal constant C such that*

$$u(x)\leq Cu(y)$$

for all $|x|=|y|\geq 2$.

Proof. Our method is scaling argument. By the Harnack inequality, we have

$$u(x)\leq Cu(y)$$

for all $|x|=|y|=2$. For $R>2$, consider $u(Rx/2)$, which is C^2, harmonic, and non-negative in $\{|x|>1\}$. Substituting it into the Harnack inequality, we have

$$u(x)\leq Cu(y)$$

for all $|x|=|y|=R$. So the lemma is proved. □

Finally, we use this lemma to prove that $u=0$. Fix x. For arbitrary $\varepsilon>0$, take R large enough such that $R>\max\{|x|,2\}$ and $\min_{|x|=R}u(x)<\varepsilon$. Then by the lemma,

$$\max_{|x|=R}u(x)\leq C\min_{|x|=R}u(x)\leq C\varepsilon.$$

Applying the maximum principle to $\{1<|x|<R\}$, we have

$$u(x)\leq C\varepsilon.$$

Let $\varepsilon\to 0$, then we obtain $u(x)=0$. So the proof is concluded.