Solutions of Homework II

1. (i) By the spherical coordinate representation of the Laplacian, let $u = f(r)$, then
$$f''(r) + \frac{2}{r}f'(r) = f(r).$$
Let $f(r) = r^{-1}g(r)$, then
$$f'(r) = \frac{1}{r}g'(r) - \frac{1}{r^2}g(r)$$
$$f''(r) = \frac{1}{r}g''(r) - \frac{2}{r^2}g'(r) + \frac{2}{r^3}g(r).$$
Therefore,
$$g''(r) = g(r).$$
So
$$g(r) = Ae^r + Be^{-r},$$
and
$$f(r) = \frac{1}{r}(Ae^r + Be^{-r}).$$

(ii) We find a spherically symmetric solution $u = f(r)$. By the spherical coordinate representation of the Laplacian,
$$f''(r) + \frac{2}{r}f'(r) = 0,$$
and $f(a) = A$ and $f(b) = B$. Then $f'(r) = C_1 r^{-2}$ and $f(r) = -C_1 r^{-1} + C_2$. By the boundary conditions,
$$f(r) = \frac{A - B}{1/a - 1/b}r^{-1} + \frac{-A/b + B/a}{1/a - 1/b}.$$
Since the solution of the Dirichlet problem is unique, the solution is the above f.

(iii) We find a spherically symmetric solution $u = f(r)$. By the spherical coordinate representation of the Laplacian,
$$f''(r) + \frac{1}{r}f'(r) = 1,$$
and $f(a) = 0$ and $f(b) = 0$. Then $f'(r) = r/2 + C_1 r^{-1}$ and $f(r) = r^2/4 +$

$C_1 \log r + C_2$. By the boundary conditions,

$$f(r) = -\frac{a^2 - b^2}{4(\log a - \log b)} \log r + \frac{a^2 \log b - b^2 \log a}{4(\log a - \log b)} + \frac{r^2}{4}.$$

Since the solution of the Dirichlet problem is unique, the solution is the above f.

(iv) We find a spherically symmetric solution $u = f(r)$. By the spherical coordinate representation of the Laplacian,

$$f''(r) + \frac{2}{r} f'(r) = 1,$$

and $f(a) = 0$ and $f(b) = 0$. Then $f'(r) = r/3 + C_1 r^{-2}$ and $f(r) = r^2/6 + C_1 r^{-1} + C_2$. By the boundary conditions,

$$f(r) = \frac{ab(a+b)}{6} r^{-1} - \frac{a^2 + ab + b^2}{6} + \frac{r^2}{6}.$$

Since the solution of the Dirichlet problem is unique, the solution is the above f.

(v) Suppose that

$$\int_D f \neq \int_{\partial D} g.$$

Since

$$\int_D \Delta u = \int_{\partial D} \frac{\partial u}{\partial n},$$

and $\Delta u = f$ and $\frac{\partial u}{\partial n} = g$, we have

$$\int_D f = \int_{\partial D} g,$$

which is contradiction.

2. (i) Suppose that $h \neq 0$. By the maximum principle, $u \geq 0$. Suppose that u vanish at some point x_0. Then by the Harnack inequality, $u(0) = 0$ and $u = 0$. Hence $h = 0$, which is contradiction.

(ii) By the maximum principle, $u \geq 0$. By the Harnack inequality,

$$\frac{1 - 1/2}{1 + 1/2} u(0) \leq u(x, y) \leq \frac{1 + 1/2}{1 - 1/2} u(0)$$

for $x^2 + y^2 = 1/4$. Therefore,

$$\frac{1}{3} \leq u(x, y) \leq 3$$

for $x^2 + y^2 = 1/4$.

3. (i) Suppose that u_1 and u_2 satisfy the equation. Let $v = u_1 - u_2$, then

$$\begin{cases} \Delta v = u_1^3 - u_2^3 & \text{in } D; \\ \dfrac{\partial v}{\partial n} + a(x)v = 0 & \text{on } \partial D. \end{cases}$$

Since

$$\int_D v\Delta v = \int_{\partial D} \frac{\partial v}{\partial n} v - \int |\nabla v|^2,$$

$$\int_D (u_1^3 - u_2^3)v = -\int_{\partial D} a(x)v^2 - \int |\nabla v|^2.$$

The left hand side is equal to

$$\int_D v^2(u_1^2 + u_1u_2 + u_2^2) \geq 0.$$

The right hand side is less than or equal to 0 because $a(x) \geq 0$. So

$$\int_D v^2(u_1^2 + u_1u_2 + u_2^2) = 0$$

and $u_1 = u_2$.

(ii) (a)

$$E[u] = \int_D \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}bu^2 + fu\right) \mathrm{d}x.$$

The admissible set is

$$\{u \in C^2(\overline{D}) \mid u = h \text{ on } \partial D\}.$$

(b) Only if: We only need to prove that for a solution u,

$$E[u + v] \geq E[u]$$

for all $v \in C^2(\overline{D})$ satisfying that $v = 0$ on ∂D.

$$\begin{aligned} E[u+v] - E[u] &= \int_D (\nabla u \cdot \nabla v + buv + fv) + \int_D \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{2}bv^2\right) \\ &\geq \int_D (\nabla u \cdot \nabla v + buv + fv). \end{aligned}$$

By integration by parts,

$$\int_D (\nabla u \cdot \nabla v + buv + fv) = \int_D (-\Delta u + bu + f)\, v + \int_{\partial D} \frac{\partial u}{\partial n} v = 0.$$

Hence

$$E[u + v] \geq E[u].$$

If: Suppose that

$$E[w] \geq E[u]$$

for all $w \in C^2(\overline{D})$ satisfying that $w = h$ on ∂D. Consider $w = u + t\eta$

where $\eta \in C_c^2(D)$, then $w \in C^2(\overline{D})$ satisfies that $w = h$ on ∂D. Therefore,

$$f(t) = E[u + t\eta] \geq E[u].$$

So

$$f'(0) = \int_D (\nabla u \cdot \nabla \eta + bu\eta + f\eta) = 0.$$

By integration by parts, it follows that

$$\int_D (-\Delta u + bu + f)\eta = 0$$

for all $\eta \in C_c^2(D)$. Hence u is a solution to the equation.

(iii) For $z = (x, y)$, let $z_l = (-x, y)$, $z_d = (x, -y)$ and $z_{ld} = (-x, -y)$. Then it is easy to see that

$$\Gamma(z, z') - \Gamma(z_l, z') - \Gamma(z_d, z') + \Gamma(z_{ld}, z')$$

is the Green's function. So

$$\int_0^\infty \frac{1}{\pi} \left(\frac{x}{x^2 + (y - y')^2} - \frac{x}{x^2 + (y + y')^2} \right) g(y')\, \mathrm{d}y' \\ + \int_0^\infty \frac{1}{\pi} \left(\frac{y}{(x - x')^2 + y^2} - \frac{y}{(x + x')^2 + y^2} \right) h(x')\, \mathrm{d}x'$$

is the solution formula for $u(x, y)$.

4. We use $u_{,i}$ to denote $\partial_i u$.

(i) It is easy to see that

$$\Delta v(x) = \Delta u(x - y).$$

So v is harmonic.

(ii) It is easy to see that

$$\Delta v(x) = \lambda^2 \Delta u(\lambda x).$$

So v is harmonic.

(iii) Suppose that $(Ox)_i = a_{ij}x_j$. Then

$$v_{,i}(x) = \sum_{k=1}^{3} u_{,k}(Ox)a_{ki};$$

$$v_{,ii}(x) = \sum_{k,l=1}^{3} u_{,kl}(Ox)a_{ki}a_{li}.$$

Since O is orthogonal,

$$OO^T = I,$$

and

$$\sum_{i=1}^{3} a_{ki}a_{li} = \delta_{kl}.$$

Therefore,

$$\Delta v(x) = \Delta u(Ox),$$

and v is harmonic.

(iv) It is easy to verify that

$$\Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f(\Delta g).$$

Note that $\Delta|x|^{-1} = 0$. We have

$$\Delta v = 2\nabla|x|^{-1} \cdot \nabla(u(x^*)) + |x|^{-1}\Delta(u(x^*)),$$

where $x^* = x/|x|^2$.

$$(u(x^*))_{,i} = \sum_{k=1}^{3} u_{,k}(x^*)(x^*)^k_{,i};$$

$$(u(x^*))_{,ii} = \sum_{k,l=1}^{3} u_{,kl}(x^*)(x^*)^k_{,i}(x^*)^l_{,i} + \sum_{k=1}^{3} u_{,k}(x^*)(x^*)^k_{,ii}.$$

$$(x^*)^i_{,j} = |x|^{-4}(\delta_{ij}|x|^2 - 2x_ix_j).$$

If we let matrix $A = ((x^*)^i_{,j})$, then

$$A = |x|^{-4}(|x|^2 I - 2xx^T),$$

where x denotes a column vector. Then

$$AA^T = |x|^{-8}(|x|^4 I - 4|x|^2 xx^T + 4|x|^2 xx^T) = |x|^{-4} I.$$

So

$$\sum_{k,l,i=1}^{3} u_{,kl}(x^*)(x^*)^k_{,i}(x^*)^l_{,i} = |x|^{-4}\Delta u(x^*) = 0.$$

To prove v is harmonic, it suffices to prove that

$$2\sum_{k,i=1}^{3} (|x|^{-1})_{,i} u_{,k}(x^*)(x^*)^k_{,i} + \sum_{k,i=1}^{3} u_{,k}(x^*)|x|^{-1}(x^*)^k_{,ii} = 0.$$

It suffices to prove that

$$\sum_{i=1}^{3} 2(|x|^{-1})_{,i}(x^*)^k_{,i} + |x|^{-1}(x^*)^k_{,ii} = 0.$$

That is,

$$2\nabla|x|^{-1} \cdot \nabla(x^*)^k + |x|^{-1}\Delta(x^*)^k = 0.$$

Since $\Delta|x|^{-1} = 0$,

$$\Delta(|x|^{-1}(x^*)^k) = 2\nabla|x|^{-1} \cdot \nabla(x^*)^k + |x|^{-1}\Delta(x^*)^k.$$

So it suffices to prove that

$$\Delta(|x|^{-1}(x^*)^k) = 0.$$

Note that

$$|x|^{-1}x^* = \frac{x}{|x|^3} = -\nabla|x|^{-1}.$$

Therefore,

$$\Delta(|x|^{-1}(x^*)^k) = 0$$

since $\Delta|x|^{-1} = 0$.

5. Let f solve

$$\begin{cases} \Delta f = \operatorname{div} \vec{E} & \text{in } B_1; \\ f = 0 & \text{on } \partial B_1. \end{cases}$$

Let $\vec{F} = \nabla f$ and $\vec{G} = \vec{E} - \vec{F}$, then it is easy to see that $\vec{E} = \vec{F} + \vec{G}$, $\operatorname{curl} \vec{F} = 0$, and $\operatorname{div} \vec{G} = 0$.