Week 8
1010a

Topics

- Indefinite integral = Differential Equation
- Notations. Terminologies
- Existence, "Uniqueness" Theorems

Terminologies.

(Differential Equations)

A Differential Equation (in the following, we simply call it "D.E.") is an "equation" (hence there must be an "equal" sign!) involving an unknown function $F(x)$ and its derivatives.

Simplest Examples of a D.E. is the following: Given a function $f(x)$, find the unknown function $F(x)$ satisfying the equation:

$$F'(x) = f(x)$$

Indefinite Integral/Primitive/Anti-derivative

The unknown function $F(x)$ is called an indefinite integral, a primitive or an anti-derivative of the given function $f(x)$.

Geometric Meaning of the D.E.

The D.E. $F'(x) = f(x)$ means the following:

On the left-hand side, we have the "slope of the tangent line to the unknown curve $y = F(x)$ at the points (x, y) (y is here "free" or "arbitrary")".

On the right-hand side, value of this slope is given by the function $f(x)$.

Using this piece of information, we can plot the "(tangent) vector field" or "(tangent) line field" of the unknown function $F(x)$. Such diagrams are called "Phase Planes" or "Phase Portraits".

Example (of a phase portrait)

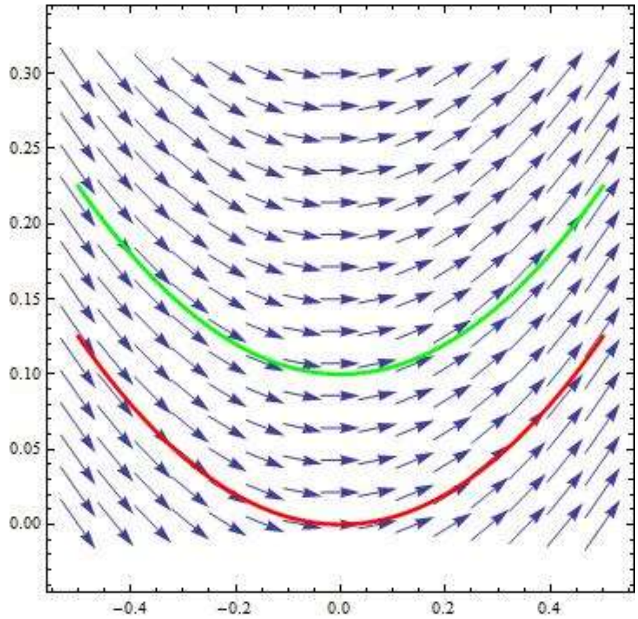
Let $f(x) = x$, then the D.E. $F'(x) = x$ says.

At the points $(0, y)$, the "slope of the tangent lines to the unknown curve $y = F(x)$" is equal to 0.

At the points $(.1, y)$, the "slope of the tangent lines to the unknown curve $y = F(x)$" is equal to 0.1.
At the points $(.2, y)$, the "slope of the tangent lines to the unknown curve $y = F(x)$" is equal to 0.2.
At the points $(-.1, y)$, the "slope of the tangent lines to the unknown curve $y = F(x)$" is equal to -0.1.
At the points $(-.2, y)$, the "slope of the tangent lines to the unknown curve $y = F(x)$" is equal to -0.2.

These statements tell us how to draw "length one" (or any other lengths) tangent lines at the given points $(0, y), (0.1, y), (0.2, y), (-0.1, y), (-0.2, y), \cdots$

From these tangent lines, one can "see" the solution curves $y = F(x)$.



One can use these diagrams to "qualitatively" understand the behavior of a D.E. without solving it. Also, one sees immediately that the solution curves of the D.E. $F'(x) = f(x)$ will not have any intersections.

The Notation $\boldsymbol{F(x) = \int f(x)dx}$

In most textbooks, instead of writing $F'(x) = f(x)$ $(*)$ the following is written:

$$F(x) = \int f(x)dx \qquad (**)$$

Explanation of (**) is equivalent to (*)

Equivalence of (*) and (**) (an intuitive explanation. Better explanation will be given later).

- Interpret $dF(x) = F'(x)dx$ as "infinitesimal change in $F(x)$ = infinitesimal change in x multiplied by the factor $F'(x)$."
- "Summing up the infinitesimal change in $F(x)$" (in symbol: $\int dF(x)$) gives back $F(x)$. (In symbol: $\int dF(x) = F(x)$)
- Using the above two bullet points, we get $F(x) = \int dF(x) = \int F'(x)dx$
- To finish the argument, note that the last term, i.e. $\int F'(x)dx$ is nothing but equal to $\int f(x)dx$ (by using $F'(x) = f(x)$). Hence $F(x) = \int f(x)dx$ is the same as $F'(x) = f(x)$.

Next, we mention a result which tells us when the equation $F'(x) = f(x)$ has solutions.

Existence Theorem

Theorem Let f be a continuous function on the closed interval $[a, b]$, then the equation $\frac{dF(x)}{dx} = f(x), x \in (a, b)$ has solutions. Also, $F(x)$ is differentiable $\forall x \in (a, b)$.

(For otherwise the term $\frac{dF(x)}{dx}$ in the D.E. has no meaning!)

As to the question "how many solutions does the equation $F'(x) = f(x)$ have, the answer is given by the

"Uniqueness" Result

The solutions of the D.E. $F'(x) = f(x)$ is not unique. But it is "unique" up to the "addition of a constant".

Theorem.

Let $f(x)$ be a continuous function on $[a, b]$. Suppose $F_1'(x) = f(x), \forall x \in (a, b)$ and $F_2'(x) = f(x), \forall x \in (a, b)$ are two "arbitrary" solutions of the D.E. $F'(x) = f(x)$. Then the difference between $F_1(x)$ and $F_2(x)$ is a constant. I.e.

$$\exists C\ \forall x \in (a, b): \ F_1(x) - F_2(x) = C.$$

Proof: Idea. Use contradiction proof. I.e. suppose the statement

$$\exists C\ \forall x \in (a, b): \ F_1(x) - F_2(x) = C$$

is false, then for the function $H(x) = F_1(x) - F_2(x)$, we have $\exists x_1, x_2 \in (a, b)$:

$$H(x_1) \neq H(x_2)$$

Hence it follows that the quotient $\frac{H(x_1)-H(x_2)}{x_1-x_2} \neq 0.$

But this quotient is equal to (by LMVT) $H'(\xi)$ $\exists \xi \in (x_1, x_2)$. Now because of our assumptions that $F_1'(x) = f(x)$ and $F_2'(x) = f(x)$, it follows that $H'(x) = F_1'(x) - F_2'(x) = 0$, $\forall x \in (a, b)$. In particular, $H'(\xi) = 0$. This is however impossible, since $0 \neq \frac{H(x_1)-H(x_2)}{x_1-x_2} = H'(\xi) = 0.$

We therefore have found a contradiction, which arose because we are assuming that $F_1(x) - F_2(x)$ is not a constant function. This means our assumption is wrong, so $F_1(x) - F_2(x)$ is a constant function.

In the following, we will use the following

Terminology

The process of finding indefinite integrals is known as "integration".

Some Simple Properties of Indefinite Integrals

The following properties of indefinite integrals are easy to check.

Theorem Let $f(x)$ and $g(x)$ be continuous functions and k be a constant. Then

1. $\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx.$
2. $\int kf(x)dx = k\int f(x)dx$, where k is a constant number.

Remark:

Note that we haven't put down the multiplication or division rules for indefinite integrals!

Simple Formulas to calculate Indefinite Integrals

All the following formulas can be checked by differentiating the right-hand side with respect to x.

1. $\int kdx = kx + C$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$
3. $\int e^x dx = e^x + C$
4. $\int \cos x\, dx = \sin x + C$
5. $\int \sin x\, dx = -\cos x + C$
6. $\int \sec^2 x\, dx = \tan x + C$
7. $\int \csc^2 x\, dx = -\cot x + C$

8. $\int \sec x \tan x \ dx = \sec x + C$
9. $\int \csc x \cot x \ dx = -\csc x + C$
10. $\int \frac{1}{x} dx = \begin{cases} \ln x + C_1, & if\ x > 0 \\ \ln(-x) + C_2, & if\ x < 0 \end{cases}$

Remark:
Item 10 above shows that theoretical knowledge about integration is sometimes important. The reason is because the function $f(x) = 1/x$ is not defined at $x = 0$, so to get the answers on the right-hand side of item 10, we have to use the existence theorem piece by piece, e.g. on any domain $[a, b]$ in the positive x −axis or and domain $[a, b]$ on the negative x −axis. Because of this, we have the formulas on the right-hand side with two different constants C_1 and C_2.