THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 8 (March 25)

Let $\emptyset \neq E \subseteq \mathbb{R}$.

Definition. A collection $\mathcal{U}$ of nondegenerate intervals is said to be a **Vitali cover** of E if for every $x \in E$, for any $\varepsilon > 0$, there is $I \in \mathcal{U}$ such that $x \in I$ and $\ell(I) < \varepsilon$.

Remark. Suppose $\mathcal{U}$ is a Vitali cover of E.

(1) Then $\{\overline{I} : I \in \mathcal{U}\}$ is also a Vitali cover of E.

(2) If G is open and $E \subseteq G$, then $\{I \in \mathcal{U} : I \subseteq G\}$ is a Vitali cover of E.

Vitali Covering Lemma. *Suppose $m^*(E) < +\infty$. Let $\mathcal{U}$ be a Vitali cover of E. Then for any $\gamma > 0$, there are disjoint $I_1, I_2, \ldots, I_N \in \mathcal{U}$ such that*

$$m^*\left(E \setminus \bigcup_{n=1}^{N} I_n\right) < \gamma.$$

Remark. (1) E need not be measurable.

(2) The result need not hold when $m^*(E) = +\infty$. For example, $\mathcal{U} := \{[x, x+r] : x \in \mathbb{R}, 0 < r < 1\}$ is a Vitali cover of $\mathbb{R}$ but $m^*(\mathbb{R} \setminus \bigcup_{n=1}^{N} I_n) = +\infty$ for any finite subcollection $\{I_1, \ldots, I_N\}$ of $\mathcal{U}$.

(3) In the proof, we actually find a countable disjoint subcollection $\{I_n\}_{n=1}^{\infty} \subseteq \mathcal{U}$ such that for some N,

$$E \subseteq \bigcup_{n=1}^{N} I_n \cup \bigcup_{n=N+1}^{\infty} \widehat{I_n}$$

and

$$\sum_{n=N+1}^{\infty} \ell(\widehat{I_n}) < \varepsilon,$$

where $\widehat{I_n}$ is the interval with the same centre as I_n and $\ell(\widehat{I_n}) = 5\ell(I_n)$.

Example 1. Let E be a union (not necessarily countable) of nondegenerate intervals (open, closed, half open and half closed, infinite, etc). Show that E is measurable.

Solution. Write $E = \bigcup_{\alpha \in \mathcal{A}} I_\alpha$, where $\mathcal{A}$ is an index set, and I_α is a nondegenerate interval. Let

$$\mathcal{U} = \{[a,b] : a < b, [a,b] \subseteq I_\alpha, \exists \alpha \in \mathcal{A}\}.$$

Then $\mathcal{U}$ is a Vitali cover of E. Let $\varepsilon > 0$. By Vitali Covering Lemma, there are disjoint $I_1, I_2, \ldots, I_N \in \mathcal{U}$ such that $m^*(E \setminus \bigcup_{n=1}^{N} I_n) < \varepsilon$. Since each I_n is closed and contained in E, so is the finite union $\bigcup_{n=1}^{N} I_n$ Now E is measurable by Littlewood's 1st principle. ◀

Vitali Covering Theorem. *Let $m^*(E) < \infty$. Let $\mathcal{U}$ be a Vitali cover of E. Then there exists a countable disjoint subcollection $\{I_n\}_{n=1}^{\infty} \subseteq \mathcal{U}$ such that*

$$m^*\left(E \setminus \bigcup_{n=1}^{\infty} I_n\right) = 0.$$

Remark. The assumption "$m^*(E) < +\infty$" can be dropped by considering the Vitali cover $\mathcal{U}_n := \{I \in \mathcal{U} : I \subseteq (n, n+1)\}$ of $E \cap (n, n+1)$ for $n \in \mathbb{Z}$.

Proof. Assume that each interval in $\mathcal{U}$ is closed. By Vitali Covering Lemma, there exist disjoint $\{I_j^{(1)} : 1 \leq j \leq m_1\} \subseteq \mathcal{U}$ such that

$$m^*\left(E \setminus \bigcup_{j=1}^{m_1} I_j^{(1)}\right) < \frac{1}{2}.$$

If $E \setminus \bigcup_{j=1}^{m_1} I_j^{(1)} \neq \emptyset$, then $\mathcal{V}_1 := \{I \in \mathcal{U} : I \subseteq \mathbb{R} \setminus \bigcup_{j=1}^{m_1} I_j^{(1)}\}$ is a Vitali cover of $E \setminus \bigcup_{j=1}^{m_1} I_j^{(1)}$, and hence, by Vitali Covering Lemma, there exist disjoint $\{I_j^{(2)} : 1 \leq j \leq m_2\} \subseteq \mathcal{V}_1$ such that

$$m^*\left((E \setminus \bigcup_{k=1}^{2}\bigcup_{j=1}^{m_k} I_j^{(k)})\right) = m^*\left((E \setminus \bigcup_{j=1}^{m_1} I_j^{(1)}) \setminus \bigcup_{j=1}^{m_2} I_j^{(2)}\right) < \frac{1}{2^2}.$$

Continue in this way, we obtain a countable disjoint subcollection

$$\{I_n\}_{n=1}^{\infty} := \{I_j^{(k)} : 1 \leq j \leq m_k,\ k \in \mathbb{N}\} \subseteq \mathcal{U}$$

such that

$$m^*\left(E \setminus \bigcup_{n=1}^{\infty} I_n\right) < \frac{1}{2^m} \qquad \text{for all } m \in \mathbb{N}.$$

Hence $m^*\left(E \setminus \bigcup_{n=1}^{\infty} I_n\right) = 0$. □

Example 2. Let $E \subseteq \mathbb{R}$, and let $f : E \to \mathbb{R}$ be a function (not necessarily measurable). For $\alpha > 0$, define

$$E_\alpha = \{x \in E : f'(x) \text{ exists and } |f'(x)| < \alpha\}.$$

Show that $m^*(f(E_\alpha)) \leq \alpha m^*(E_\alpha)$.

Solution. We may assume that $m^*(E_\alpha) < +\infty$. Let $\varepsilon > 0$. Choose an open $G \supseteq E_\alpha$ such that $m(G) < m^*(E_\alpha) + \varepsilon$. Note that for any $x \in E_\alpha$, there is $\delta_x > 0$ such that if $0 < r < \delta_x$, then

$$|f(y) - f(x)| < \alpha|y - x| \qquad \text{for any } y \in B(x, r),$$

so that

$$f(B(x, r)) \subseteq B(f(x), \alpha r). \tag{\#}$$

Let

$$\mathcal{U} = \{B(x, r) : x \in E_\alpha,\ 0 < 5r < \delta_x,\ B(x, r) \subseteq G\}.$$

Then $\mathcal{U}$ is a Vitali cover of E_α. By Remark (3) of Vitali Covering Lemma, there is a countable disjoint subcollection $\{B(x_n, r_n)\}_{n=1}^{\infty} \subseteq \mathcal{U}$ such that

$$E_\alpha \subseteq \bigcup_{n=1}^{N} B(x_n, r_n) \cup \bigcup_{n=N+1}^{\infty} B(x_n, 5r_n)$$

and $\displaystyle\sum_{n=N+1}^{\infty} m(B(x_n, 5r_n)) < \varepsilon$, for some N. Now (**??**) yields

$$\begin{aligned} f(E_\alpha) &\subseteq \bigcup_{n=1}^{N} f(B(x_n, r_n)) \cup \bigcup_{n=N+1}^{\infty} f(B(x_n, 5r_n)) \\ &\subseteq \bigcup_{n=1}^{N} B(f(x_n), \alpha r_n) \cup \bigcup_{n=N+1}^{\infty} B(f(x_n), 5\alpha r_n). \end{aligned}$$

Hence, by the scaling property of m, and the disjointness of $\{B(x_n, r_n)\}$,

$$\begin{aligned} m^*(f(E_\alpha)) &\leq \sum_{n=1}^{N} m^*\left(B(f(x_n), \alpha r_n)\right) + \sum_{n=N+1}^{\infty} m^*\left(B(f(x_n), 5\alpha r_n)\right) \\ &= \alpha \sum_{n=1}^{N} m\left(B(x_n, r_n)\right) + \alpha \sum_{n=N+1}^{\infty} m\left(B(x_n, 5r_n)\right) \\ &\leq \alpha m\left(\bigcup_{n=1}^{N} B(x_n, r_n)\right) + \alpha\varepsilon \\ &\leq \alpha m\left(G\right) + \alpha\varepsilon \\ &\leq \alpha m\left(E_\alpha\right) + 2\alpha\varepsilon. \end{aligned}$$

The result follows since $\varepsilon > 0$ is arbitrary. ◀