THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 5 (March 4)

Recall that $\mathcal{M}$ is the σ-algebra of all measurable subsets of $\mathbb{R}$.

Definition. Let $E \in \mathcal{M}$, $f\colon E \to [-\infty, \infty] =: \mathbb{R}^*$ be a function. Then f is said to be measurable if $\{x \in E : f(x) > \alpha\} \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$.

Proposition 1. The following are equivalent.

(i) $\{x \in E : f(x) > \alpha\} \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$;

(ii) $\{x \in E : f(x) \geq \alpha\} \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$;

(iii) $\{x \in E : f(x) < \alpha\} \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$;

(iv) $\{x \in E : f(x) \leq \alpha\} \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$.

Remark. (1) The statements (i)–(iv) imply (v) $\{x \in E : f(x) = \alpha\} \in \mathcal{M}$ for all $\alpha \in \mathbb{R}^*$. However the converse is not true. For example, let $P \subseteq [0, 1]$ be a non-measurable set and define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in P, \\ -x & \text{if } x \in [0,1] \setminus P. \end{cases}$$

Then $f^{-1}(\{\alpha\}) = \{\alpha\}, \{-\alpha\}$ or $\emptyset$ for all $\alpha \in \mathbb{R}$; $f^{-1}(\{\pm\infty\}) = \emptyset$. So f satisfies (v). However f is not measurable since $f^{-1}((0, \infty)) = P \notin \mathcal{M}$.

(2) f is measurable if and only if $f^{-1}(B) \in \mathcal{M}$ for all Borel $B \subseteq \mathbb{R}$.

Proposition 2. Let $E \in \mathcal{M}$, let $f, g\colon E \to \mathbb{R}$. Suppose $c \in \mathbb{R}$. Then $f + c, cf, f \pm g, fg, f^2, f \vee g, f \wedge g$ are all measurable.

Remark. The same is true even if f, g are $\mathbb{R}^*$-valued. The values $\pm\infty$ need to be treated separately.

Example 1. Let $D \in \mathcal{M}$ and $f\colon \ \to \mathbb{R}^*$. Set $D_1 = \{x : f(x) = +\infty\}$ and $D_2 = \{x : f(x) = -\infty\}$. Show that f is measurable if and only if $D_1, D_2 \in \mathcal{M}$ and $f\big|_{D \setminus (D_1 \cup D_2)}$ is measurable.

Example 2. Let $D \in \mathcal{M}$ and let $f, g\colon D \to \mathbb{R}^*$ be measurable functions. Show that fg is also measurable.

Example 3. Show that the composition of two measurable functions may not be measurable.

Solution. In the last tutorial, we showed that there is a continuous, strictly increasing function ψ that maps a non-measurable set P to a measurable set $\psi(P)$. Now $f := \chi_{\psi(P)}$ is measurable, and $g := \psi$ is continuous hence measurable. However $f \circ g = \chi_P$ is not measurable. ◀

The Littlewood's Three Principles can be roughly expressed as follows:

Littlewood's 1st Principle. *'Every' set is nearly a finite union of intervals.*

Littlewood's 2nd Principle. *'Every' function is nearly continuous.*

Littlewood's 3rd Principle. *'Every' convergent sequence of functions is nearly uniformly convergent.*

Example 4 (Lusin's Theorem)**.** Let $E \in \mathcal{M}$ with $m(E) < +\infty$. Let $f \colon E \to \mathbb{R}$ be measurable. Show that for any $\varepsilon > 0$, there is a closed set $F \subseteq E$ such that $m(E \setminus F) < \varepsilon$ and $f\big|_F$ is continuous.