THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 3 (February 19)

Definition (Lower semi-continuity)**.** Let $f : A \to \mathbb{R}$ and $x_0 \in A$. f is said to be lower semi-continuous (l.s.c) at x_0 if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x_0) - \varepsilon < f(x) \qquad \text{for all } x \in A \cap V_\delta(x_0).$$

Example 1 (HW3 Q1)**.** Let $f : [a, b] \to \mathbb{R}$. The Lower Envelope of f is the function $\underline{f} : [a, b] \to [-\infty, \infty]$ defined by

$$\underline{f}(x) := \sup\{g_\delta(x) : \delta > 0\} \quad \text{for all } x \in [a, b],$$

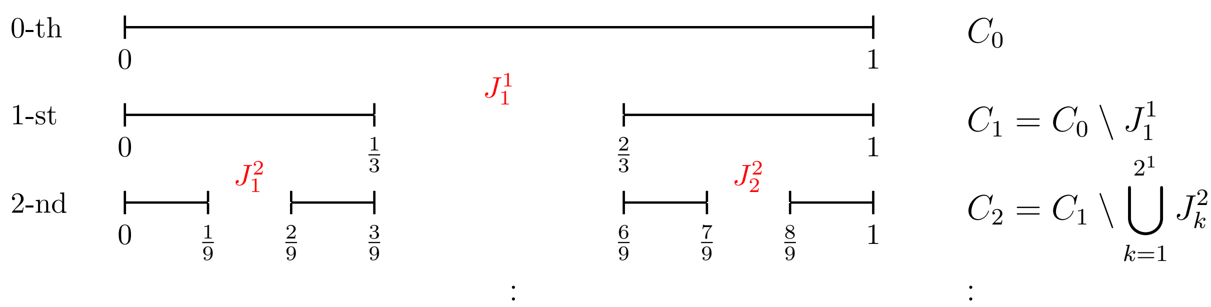
where $g_\delta(x) := \inf\{f(y) : y \in [a, b] \cap V_\delta(x)\}$.

(a) Let $x \in [a, b]$. Show that $\underline{f}(x) \le f(x)$, and $\underline{f}(x) = f(x)$ if and only if f is l.s.c at x.

(b) Show that if f is bounded, then $\underline{f}$ is l.s.c.

(c) Show that if $\phi\colon [a, b] \to \mathbb{R}$ is l.s.c on $[a, b]$ and $\phi \le f$, then $\phi \le \underline{f}$.

Example 2 (The Cantor set)**.** Consider the following construction:



Note that we remove 2^n open intervals of length 3^{-n-1} at the $(n + 1)$-th step.

In this way, we obtain a sequence of sets $\{C_n\}$ defined by

$$C_0 = [0, 1], \qquad C_{n+1} = C_n \setminus \bigcup_{k=1}^{2^n} J_k^{n+1},$$

where $\{J_k^{n+1} : k = 1, \ldots, 2^n\}$ are the middle-1/3 open intervals removed from each of the closed bounded intervals in C_n. The sequence $\{C_n\}$ is decreasing, nonempty and compact. By Cantor's intersection Theorem, $\bigcap_{n=1}^{\infty} C_n$ is nonempty. We write $\mathcal{C} := \bigcap_{n=1}^{\infty} C_n$ and call it the Cantor set.

The Cantor set $\mathcal{C}$ satisfies the following properties.

(a) $\mathcal{C}$ is compact.

(b) $\mathcal{C}$ is perfect (hence uncountable). (Recall that a set A is perfect if every point in A is an accumulation point of A.)

(c) $m(\mathcal{C}) = 0$.

Solution. (a) Clear.

(b) Suppose $x \in \mathcal{C}$. Then $x \in C_n$ for all n. Let $I_n = [a_n, b_n]$ be the interval in C_n that contains x. It is clear from the construction of $\mathcal{C}$ that $a_n, b_n \in \mathcal{C}$ and $|a_n - b_n| = 3^{-n}$. Hence $\lim(a_n) = \lim(b_n) = x$. Thus x is an accumulation point of $\mathcal{C}$.

(c) Note that

$$C_{n+1} = C_n \setminus \bigcup_{k=1}^{2^n} J_k^{n+1} = \left(C_{n-1} \setminus \bigcup_{k=1}^{2^{n-1}} J_k^n \right) \setminus \bigcup_{k=1}^{2^n} J_k^{n+1} = \cdots = C_0 \setminus \bigcup_{m=0}^{n} \bigcup_{k=1}^{2^m} J_k^{m+1}.$$

Hence

$$\begin{aligned} m(C_{n+1}) &= m([0,1]) - \sum_{m=0}^{n} \sum_{k=1}^{2^m} m(J_k^{m+1}) \\ &= 1 - \sum_{m=0}^{n} \sum_{k=1}^{2^m} \frac{1}{3^{m+1}} \\ &= 1 - \frac{1}{3} \sum_{m=0}^{n} \left(\frac{2}{3} \right)^m . \end{aligned}$$

By Monotone Convergence Lemma for measures,

$$m(\mathcal{C}) = \lim m(C_{n+1}) = 1 - \frac{1}{3} \sum_{m=0}^{\infty} \left(\frac{2}{3} \right)^m = 1 - \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 0.$$

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