THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 1 (February 12)

Definition. The *Borel σ-algebra* $\mathcal{B}$ is the smallest σ-algebra which contains all open subsets of $\mathbb{R}$, that is

$$\mathcal{B} := \bigcap \{\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}) : \mathcal{A} \text{ is a } \sigma\text{-algebra containing all open subsets of } \mathbb{R}\}$$

The members of $\mathcal{B}$ are called *Borel sets*.

Remark. (1) We also say that $\mathcal{B}$ is the σ-algebra generated by the open sets in $\mathbb{R}$ and write $\mathcal{B} = \sigma(G)$, where G is the collection of all open subsets of $\mathbb{R}$.

(2) Since any open set in $\mathbb{R}$ can be expressed as a countable (disjoint) union of open intervals and $(a, b) = \bigcup_{n=1}^{\infty}[a + 1/n, b - 1/n]$, we have

$$\mathcal{B} = \sigma(\{(a, b) : a < b\}) = \sigma(\{[a, b] : a < b\}).$$

Definition. (i) A set is G_δ if it is a countable intersection of open sets.

(ii) A set is F_σ if it is a countable union of closed sets.

(iii) A set is $G_{\delta\sigma}$ if it is a countable union of G_δ-sets.

(iv) A set is $F_{\sigma\delta}$ if it is a countable intersection of F_σ-sets.

Remark. (1) An open set is F_σ and a closed set is G_δ.

(2) $G \subseteq G_\delta \subseteq G_{\delta\sigma} \subseteq \cdots \subseteq \mathcal{B}$ and $F \subseteq F_\sigma \subseteq F_{\sigma\delta} \subseteq \cdots \subseteq \mathcal{B}$.

Example 1. Show that a finite union or intersection of G_δ set is G_δ. The same result holds for $F_\sigma, G_{\delta\sigma}, F_{\sigma\delta}$-sets, and so on.

Solution. It suffices to consider the union and intersection of two sets. Let G, G' be G_δ-sets. Then there exists sequences $(O_n)_{n\in\mathbb{N}}$, $(O'_m)_{m\in\mathbb{N}}$ of open sets such that

$$G = \bigcap_{n\in\mathbb{N}} O_n \quad \text{and} \quad G' = \bigcap_{m\in\mathbb{N}} O'_m.$$

Now

$$\begin{aligned} G \cup G' &= \left(\bigcap_{n\in\mathbb{N}} O_n\right) \cup G' = \bigcap_{n\in\mathbb{N}} (O_n \cup G') = \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} (O_n \cup O'_m) \\ &= \underbrace{\bigcap_{(n,m)\in\mathbb{N}\times\mathbb{N}}}_{\text{countable intersection}} \underbrace{(O_n \cup O'_m)}_{\text{open}}. \end{aligned}$$

Therefore $G \cup G'$ is G_δ. It is obvious that $G \cap G'$ is also G_δ. ◀

Example 2. Give an example for each of the following:

(a) An F_σ-set that is not G_δ.

(b) A Borel set that is neither F_σ nor G_δ.

Solution. (a) Clearly $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$ is F_σ.

Suppose $\mathbb{Q} = \bigcap_n O_n$ where each O_n is open. Then $C_n := \widetilde{O_n}$ is closed nowhere dense since O_n is open dense. Now

$$\mathbb{R} = \mathbb{Q} \cup \widetilde{\mathbb{Q}} = \bigcup_{q\in\mathbb{Q}}\{q\} \cup \bigcup_n C_n,$$

which is a countable union of closed nowhere dense sets, contradicting the Baire Category Theorem.

(b) By the same argument in (a), one can show that $E := \mathbb{Q} \cap [0,\infty)$ is Borel but not G_δ. By considering complement, we have that $F := \widetilde{\mathbb{Q}} \cap (-\infty, 0]$ is Borel but not F_σ. Now $E \cup F$ is Borel but neither G_δ nor F_σ.

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Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and B be a Borel subset of $\mathbb{R}$. Show that $f^{-1}(B)$ is Borel.

Solution. Let

$$\mathcal{A} = \{A \in \mathcal{P}(\mathbb{R}) : f^{-1}(A) \in \mathcal{B}\}.$$

If we can show that $\mathcal{A}$ is a σ-algebra that contains all open sets, then $\mathcal{B} \subseteq \mathcal{A}$ since $\mathcal{B}$ is the smallest such σ-algebra.

(I) $\mathcal{A}$ is a σ-algebra:

(a) $f^{-1}(\emptyset) = \emptyset \in \mathcal{B} \implies \emptyset \in \mathcal{A}$;

(b) $A \in \mathcal{A} \implies f^{-1}(\widetilde{A}) = \widetilde{f^{-1}(A)} \in \mathcal{B} \implies \widetilde{A} \in \mathcal{A}$;

(c) $(A_n) \subseteq \mathcal{A} \implies f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n) \in \mathcal{B} \implies \bigcup_n A_n \in \mathcal{A}$.

(II) $\mathcal{A}$ contains all open sets:

By the continuity of f, $\forall$ open $O \subseteq \mathbb{R}$, $f^{-1}(O)$ is open, hence Borel. So A contains all open sets.

Thus $\mathcal{B} \subseteq \mathcal{A}$, which means $f^{-1}(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$. ◀

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be an **injective** continuous function, and B be a Borel subset of $\mathbb{R}$. Show that $f(B)$ is Borel.

Solution. Let $\mathcal{C} = \{A \in \mathcal{P}(\mathbb{R}) : f(A) \in \mathcal{B}\}$.

(I) $\mathcal{C}$ is a σ-algebra:

- (a) $f(\mathbb{R}) = f(\bigcup_n [-n, n]) = \bigcup_n \overbrace{f([-n, n])}^{\text{compact}} \in \mathcal{B} \implies \mathbb{R} \in \mathcal{C}$;
- (b) f injective $\implies$ $f(\mathbb{R}) = f(A) \underset{\circ}{\bigcup} f(\widetilde{A}) \implies f(\widetilde{A}) = f(\mathbb{R}) \setminus f(A)$. So $A \in \mathcal{C} \implies \widetilde{A} \in \mathcal{C}$;
- (c) $(A_n) \subseteq \mathcal{C} \implies f(\bigcup_n A_n) = \bigcup_n f(A_n) \in \mathcal{B} \implies \bigcup_n A_n \in \mathcal{C}$.

(II) $\mathcal{C}$ contains all *closed bounded intervals*:

By the continuity of f, $\forall\, a < b$, $f[a, b]$ is compact, hence Borel. So $\mathcal{C}$ contains all closed bounded intervals, hence all open sets.

Thus $\mathcal{B} \subseteq \mathcal{C}$, which means $f(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$. ◀

Example 5. Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions. Show that the set of points where (f_n) converges to a finite limit is an $F_{\sigma\delta}$-set.