THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Suggested Solution of Homework 17JAN2020

1. Show that, $\forall \emptyset \neq A \subseteq B \subseteq \mathbb{R}^*$,

   (a) $\sup A \leq \sup B$

   (b) $\inf A \geq \inf B$

   (c) $\sup(A+B) \leq \sup A + \sup B$

   (d) $\inf(A+B) \geq \inf A + \inf B$

   (e) $\sup(-A) = -\inf A$.

   Solution. (a) For any $a \in A$, we have $a \in B$, and hence $a \leq \sup B$. Thus $\sup A \leq \sup B$.

   (b) Similar to (a).

   (c) For $a \in A$, $b \in B$, we have $a \leq \sup A$, $b \leq \sup B$, so that $a+b \leq \sup A + \sup B$. Thus $\sup A + \sup B$ is an upper bound of $A+B$, and hence

   $$\sup(A+B) \leq \sup A + \sup B.$$

   (d) Similar to (c).

   (e) For any $a \in A$, we have $a \geq \inf A$, which implies that $-a \leq -\inf A$. Therefore $\sup(-A) \leq -\inf A$.

   Similarly, for any $a \in A$, we have $-a \leq \sup(-A)$, which implies that $a \geq -\sup(-A)$. Thus $\inf A \geq -\sup(-A)$, that is, $\sup(-A) \geq -\inf A$.

   ◀

2. Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of sets and $B_n := A_n \setminus \left(\bigcup_{i<n} A_i\right)$ $\forall n > 1$. Show (by the well-order principle) that $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$.

   Solution. Let $B_1 = A_1$. Suppose $\bigcup_{n=1}^{k} B_n = \bigcup_{n=1}^{k} A_n$ for some $k \geq 1$. Then

   $$\begin{aligned}\bigcup_{n=1}^{k+1} B_n &= \bigcup_{n=1}^{k} B_n \cup B_{k+1} \\ &= \bigcup_{n=1}^{k} A_n \cup \left(A_{k+1} \setminus (\bigcup_{i<k+1} A_i)\right) \\ &= \bigcup_{n=1}^{k+1} A_n.\end{aligned}$$

By induction, $\bigcup_{n=1}^{k} B_n = \bigcup_{n=1}^{k} A_n$ for all $k \in \mathbb{N}$. Hence $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$. ◀

3. Let $f\colon A \to \mathbb{R}$ (A, for simplicity an interval) and $x_0 \in A$. We say that f is lower semicontinuous (lsc) at x_0 if $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t.

$$f(x_0) - \varepsilon < f(x) \qquad \forall x \in A \cap V_\delta(x_0).$$

Show that (i) $\iff$ (ii) $\iff$ (iii), where

(i) f is lsc at x_0

(ii) $f(x_0) \le \sup_{\delta>0} \inf_{u\in A\cap V_\delta(x_0)} f(u)$

(iii) $f(x_0) \le \sup_{\delta>0} \inf_{u\in (A\setminus\{x_0\})\cap V_\delta(x_0)} f(u)$.

Solution. (i) $\implies$ (ii). Suppose f is lsc at x_0. Then it follows immediately from the definition that $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $f(x_0) - \varepsilon \le \inf_{u\in A\cap V_\delta(x_0)} f(u)$. Hence

$$f(x_0) - \varepsilon \le \sup_{\delta>0} \inf_{u\in A\cap V_\delta(x_0)} f(u).$$

As $\varepsilon > 0$ is arbitrary, (ii) follows.

(ii) $\implies$ (iii). It is clear since $\inf_{u\in A\cap V_\delta(x_0)} f(u) \le \inf_{u\in (A\setminus\{x_0\})\cap V_\delta(x_0)} f(u)$ for any $\delta > 0$.

(iii) $\implies$ (i). Assume (iii) holds and let $\varepsilon > 0$. By the definition of supremum, $\exists \delta > 0$ such that

$$f(x_0) - \varepsilon < \inf_{u\in (A\setminus\{x_0\})\cap V_\delta(x_0)} f(u).$$

Clearly, $f(x_0) - \varepsilon < f(x_0)$. We have

$$f(x_0) - \varepsilon < f(u) \qquad \forall u \in A \cap V_\delta(x_0).$$

Thus f is lsc at x_0.

◀

4. Let $(X, \mathcal{A}, \mu)$ be a "measure space": X is a set, $\mathcal{A}$ a σ-algebra of subsets of X, and $\mu\colon \mathcal{A} \to [0, +\infty]$ a measure. Show that

(a) If $A \subseteq B$ and $\mu(A) < +\infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(b) Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ with $A_n \subseteq A_{n+1}\ \forall n$. Show that $\mu(A_n) \le \mu(A_{n+1})\ \forall n$ and $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_n \mu(A_n)$.

(c) Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ with $A_n \supseteq A_{n+1}\ \forall n$. Show that $\lim_n \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)$, provided that $\mu(A_N) < +\infty$ for some $N \in \mathbb{N}$.

Solution. (a) Write $B = A \cup (B \setminus A)$. Since A and $B \setminus A$ are disjoint sets in $\mathcal{A}$, it follows from the additivity of measure that

$$\mu(B) = \mu(A) + \mu(B \setminus A).$$

As $\mu(A) < +\infty$, we have $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(b) By the same argument in (a), we have

$$\mu(A_{n+1}) = \mu(A_n) + \mu(A_{n+1} \setminus A_n) \geq \mu(A_n) \qquad \text{for all } n \in \mathbb{N}.$$

Let $B_n \coloneqq A_n \setminus \left(\bigcup_{i<n} A_i \right)$ $\forall n \geq 1$. Then $\{B_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets in $\mathcal{A}$ such that

$$A_N = \bigcup_{n=1}^{N} B_n \quad \forall N \geq 1 \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Hence, by the countable additivity of μ, we have

$$\begin{aligned} \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} B_n) &= \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) = \lim_{N\to\infty} \mu(\bigcup_{n=1}^{N} B_n) \\ &= \lim_{N\to\infty} \mu(A_N). \end{aligned}$$

(c) For $n \geq N$, define $C_n = A_N \setminus A_n$. Then $C_n \subseteq C_{n+1}$ for $n \geq N$ and

$$\bigcup_{n=N}^{\infty} C_n = A_N \setminus \left(\bigcap_{n=N}^{\infty} A_n \right)$$

Note that $\mu(\bigcap_{n=N}^{\infty} A_n) \leq \mu(A_N) < +\infty$. By (a) and (b), we have

$$\mu(A_N) - \mu(\bigcap_{n=N}^{\infty} A_n) = \mu(\bigcup_{n=N}^{\infty} C_n) = \lim_n \mu(C_n) = \mu(A_N) - \lim_n \mu(A_n).$$

As $\{A_n\}_{n=1}^{\infty}$ is decreasing, we obtain

$$\lim_n \mu(A_n) = \mu(\bigcap_{n=N}^{\infty} A_n) = \mu(\bigcap_{n=1}^{\infty} A_n).$$

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5. In $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$, show the "Generalized" Monotone Convergence Theorem for sequences of extended-real numbers: If (a_n) is a monotone sequence of extended-real numbers, then it converges to a limit in $\mathbb{R}^*$. Show further that

$$\limsup x_n \coloneqq \inf_{k\in\mathbb{N}} (\sup_{n\geq k} x_n)$$

$$\liminf x_n \coloneqq \sup_{k\in\mathbb{N}} (\inf_{n\geq k} x_n)$$

exist in $\mathbb{R}^*$, and that

$$\liminf x_n = \limsup x_n \quad \text{iff} \quad \lim_n x_n \text{ exists}$$

(and all three are the same then).

Solution. To prove the "Generalized" Monotone Convergence Theorem, it suffices to show that an unbounded increasing sequence (a_n) converges to $+\infty$. Let $\alpha > 0$. Clearly a_1 is a lower bound of (a_n). As (a_n) is unbounded, α is not an upper bound. So there is $N \in \mathbb{N}$ such that $a_N > \alpha$. Since (a_n) is increasing, we have $a_n \geq a_N > \alpha$ for all $n \geq N$. Hence $\lim(a_n) = +\infty = \sup_n a_n$.

By our extended definitions of supremum and infimum for subsets of $\mathbb{R}^*$, $\limsup x_n$ and $\liminf x_n$ clearly exist in $\mathbb{R}^*$. Furthermore, it follows from the "Generalized" Monotone Convergence Theorem that the decreasing sequence $(\sup_{n\geq k} x_n)_{k=1}^{\infty}$ and the increasing sequence $(\inf_{n\geq k} x_n)_{k=1}^{\infty}$ both converge (in $\mathbb{R}^*$) with limits, respectively,

$$\lim_k(\sup_{n\geq k} x_n) = \inf_{k\in\mathbb{N}}(\sup_{n\geq k} x_n) \quad \text{and} \quad \lim_k(\inf_{n\geq k} x_n) = \sup_{k\in\mathbb{N}}(\inf_{n\geq k} x_n). \tag{$*$}$$

($\Longrightarrow$). Suppose $\liminf x_n = \limsup x_n = \ell$. If $\ell \in \mathbb{R}$, then for any $\varepsilon > 0$, we have $\limsup x_n < \ell + \varepsilon$ and $\liminf x_n > \ell - \varepsilon$. Hence there exits $k \in \mathbb{N}$ such that

$$x_n \leq \sup_{n\geq k} x_n < \ell + \varepsilon \qquad \text{for all } n \geq k,$$

and

$$x_n \geq \inf_{n\geq k} x_n > \ell - \varepsilon \qquad \text{for all } n \geq k.$$

Combining two inequalities above, we have $\lim_n x_n = \ell$.

If $\ell = +\infty$, then for any $\alpha > 0$, there exits $k \in \mathbb{N}$ such that $x_n \geq \inf_{n\geq k} x_n > \alpha$ for all $n \geq k$. Thus $\lim_n x_n = +\infty$. The proof is similar for $\ell = -\infty$.

($\Longleftarrow$). Suppose $\lim_n x_n = \ell$. If $\ell \in \mathbb{R}$, then for any $\varepsilon > 0$, there exits $N \in \mathbb{N}$ such that $\ell - \varepsilon < x_n < \ell + \varepsilon$ for $n \geq N$. Thus

$$\ell - \varepsilon \leq \inf_{n\geq k} x_n \leq \sup_{n\geq k} x_n \leq \ell + \varepsilon \qquad \text{for all } k \geq N.$$

Letting $k \to \infty$, it follows from ($*$) that

$$\ell - \varepsilon \leq \liminf x_n \leq \limsup x_n \leq \ell + \varepsilon.$$

Since ε is arbitrary, we have $\liminf x_n = \limsup x_n = \ell$.

The cases $\ell = \pm\infty$ can be proved in similar fashions. ◀