MATH 4050 Real Analysis

Suggested Solution of Homework 7

Only the solutions to $*$ questions are provided.

2.* (3rd: P.89, Q4; 4th: P.89, Q24)
Let f be a nonnegative measurable function.

(a) Show that there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions each of which vanishes outside a set of finite measure such that $f = \lim \varphi_n$.

(b) Show that $\int f = \sup \int \varphi$ over all nonnegative simple functions $\varphi \leq f$ with φ vanishes outside a set of finite measure.

Solution. (a) For each $n \in \mathbb{N}$, let

$$A_n = f^{-1}[n, \infty], \quad \text{and} \quad B_{n,k} = f^{-1}[(k-1)2^{-n}, k2^{-n}) \text{ for } k = 1, 2, \ldots, n2^n.$$

Define

$$\varphi_n = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \chi_{B_{n,k} \cap [-n,n]} + n\chi_{A_n \cap [-n,n]},$$

Then each φ_n is a nonnegative simple function that vanishes outside a set of finite measure. Moreover, $\varphi_n \leq \varphi_{n+1} \leq f$ for all n. To see that $f = \lim \varphi_n$, note that if $f(x) < \infty$, then

$$0 \leq f(x) - \varphi_n(x) \leq 2^{-n} \quad \text{for all } n \geq \lceil f(x) \rceil,$$

so that $\lim \varphi_n(x) = f(x)$; while if $f(x) = \infty$, then, for $n \geq |x|$, $\varphi_n(x) = n \to \infty$.

(b) From the monotonicity of integral, it is clear that $\int f \geq \sup \int \varphi$ over all simple functions $\varphi \leq f$ that vanishes outside a set of finite measure. Let $\{\varphi_n\}$ be the sequence of simple functions defined in (a). The Monotone Convergence Theorem implies that

$$\int f = \lim_n \int \varphi_n = \sup_n \int \varphi_n,$$

so that $\int f \leq \sup \int \varphi$ over all simple functions $\varphi \leq f$ that vanishes outside a set of finite measure.

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3.* (3rd: P.89, Q5)
Let f be a nonnegative integrable function. Show that the function F defined by

$$F(x) = \int_{-\infty}^{x} f$$

is continuous by the monotone convergence theorem.

Solution. Let $c \in \mathbb{R}$. Let $\{s_n\}$ be a sequence of real numbers increasing to c. Set $f_n := f\chi_{(-\infty,s_n)}$ for each n. Then $\{f_n\}$ is a sequence nonnegative measurable functiosn such that $f_n \uparrow f\chi_{(-\infty,c)}$. By the Monotone Convergence Theorem,

$$\lim_n F(s_n) = \lim_n \int f_n = \int f\chi_{(-\infty,c)} = F(c).$$

Similarly, if $\{t_n\}$ is a sequence of real numbers decreasing to c, set $g_n := f - f\chi_{(-\infty,t_n)}$ for each n. Then $\{g_n\}$ is a sequence nonnegative measurable functions such that $g_n \uparrow f - f\chi_{(-\infty,c]}$. By the Monotone Convergence Theorem and that $\displaystyle\int f < \infty$, we have

$$\int f - \lim_n F(t_n) = \lim_n \int g_n = \int f - F(c),$$

and hence $\lim_n F(t_n) = F(c)$.

As $\{s_n\}$ and $\{t_n\}$ are arbitrary, F is continuous at c. ◀

4.* (3rd: P.89, Q6; 4th: P.89, Q25)
Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converges to f, and suppose $f_n \leq f$ for each n. Show that

$$\int f = \lim \int f_n.$$

Solution. As $f_n \leq f$ for each n, we have $\int f_n \leq \int f$ and hence

$$\limsup_n \int f_n \leq \int f.$$

On the other hand, Fatou's Lemma implies that

$$\int f \leq \liminf_n \int f_n.$$

Thus we must have

$$\liminf_n \int f_n = \limsup_n \int f_n = \int f,$$

that is $\displaystyle\int f = \lim \int f_n$. ◀