MATH 4050 Real Analysis

Suggested Solution of Homework 4 (additional)

Only the solution to the last question is provided.

9.* Let $0 < m(E) < \infty$, $\alpha \in (1/2, 1)$ and I an open interval such that $m(E_0) > \alpha \cdot \ell(I)$, where $E_0 \coloneqq E \cap I$. Find $\delta > 0$ such that $V_\delta(0) \subseteq E_0 - E_0$.

Solution. Take $\delta = (2\alpha - 1)\ell(I)$. Let $v \in V_\delta(0)$. Then $E_0 \cup (E_0 + v)$ is contained in an interval of length at most

$$\ell(I) + \delta = (1 + (2\alpha - 1))\ell(I) = 2\alpha\ell(I).$$

If E_0 and $E_0 + v$ are disjoint, then

$$2m(E_0) = m(E_0) + m(E_0 + v) = m(E_0 \cup_0 (E_0 + v)) \leq 2\alpha\ell(I),$$

contradicting $m(E_0) > \alpha\ell(I)$. Therefore $E_0 \cap (E_0 + v) \neq \emptyset$, so that there are $a, b \in E_0$ such that $a = b + v$. Hence $v = a - b \in E_0 - E_0$.

◀