MATH 4050 Real Analysis

Suggested Solution of Homework 4

Only the solutions to $*$ questions are provided.

1.* (3rd: P.64, Q9)
Show that if E is a measurable set, then each translate $E+y$ of E is also measurable.

Solution. Let $A \subseteq \mathbb{R}$. Then $A - y \subseteq \mathbb{R}$ and the translation invariance of m^* yields

$$\begin{aligned} m^*(A) &= m^*(A-y) = m^*\big((A-y)\cap E\big) + m^*\big((A-y)\cap \tilde{E}\big) \\ &= m^*\Big(\big((A-y)\cap E\big)+y\Big) + m^*\Big(\big((A-y)\cap \tilde{E}\big)+y\Big) \\ &= m^*\big(A\cap(E+y)\big) + m^*\big(A\cap(E+y)^{\sim}\big). \end{aligned}$$

Hence $E+y$ is also measurable. ◀

2.* (3rd: P.64, Q10; 4th: P.43, Q24)
Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Solution. If either $m(E_1)$ or $m(E_2)$ is infinite, the equality is trivial. Suppose $m(E_1), m(E_2) < \infty$. Since $E_1 \cap E_2 \subset E_2$ and it is measurable with $m(E_1 \cap E_2) \leq m(E_2) < \infty$, we have

$$m(E_2 \setminus (E_1 \cap E_2)) = m(E_2) - m(E_1 \cap E_2).$$

Write $E_1 \cup E_2$ as a disjoint union, $E_1 \cup E_2 = E_1 \cup_0 (E_2 \setminus (E_1 \cap E_2))$. Then

$$m(E_1 \cup E_2) = m(E_1) + m(E_2 \setminus (E_1 \cap E_2)) = m(E_1) + m(E_2) - m(E_1 \cap E_2),$$

that is, $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$. ◀

3. (3rd: P.64, Q11; 4th: P.43, Q25)
Show that the condition $m(E_1) < \infty$ is necessary in Proposition 14 (3rd ed.) by giving a decreasing sequence $\{E_n\}$ of measurable sets with $\emptyset = \bigcap E_n$ and $m(E_n) = \infty$ for each n.

Proposition 14: Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets, that is, a sequence with $E_{n+1} \subset E_n$ for each n. Let $m(E_1)$ be finite. Then

$$m\Big(\bigcap_{i=1}^{\infty} E_i\Big) = \lim_{n\to\infty} m(E_n).$$

4.* (3rd: P.70, Q21; 4th: P.59, Q2,6)

(a) Let D and E be measurable sets and f a function with domain $D \cup E$. Show that f is measurable if and only if its restrictions to D and E are measurable.

(b) Let f be a function with measurable domain D. Show that f is measurable if and only if the function g defined (on $\mathbb{R}$) by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Solution. (a) ($\Rightarrow$): Suppose f is measurable. Then

$$\{x \in D : f\big|_D(x) > \alpha\} = \{x \in D \cup E : f(x) > \alpha\} \cap D$$

is also measurable. Similarly $\{x \in E : f\big|_E(x) > \alpha\}$ is measurable. Since $\alpha \in \mathbb{R}$ is arbitrary, both restrictions $f\big|_D$ and $f\big|_E$ are measurable.

($\Leftarrow$): It follows immediately from the following equation:

$$\{x \in D \cup E : f(x) > \alpha\} = \{x \in D : f\big|_D(x) > \alpha\} \cup \{x \in E : f\big|_E(x) > \alpha\}.$$

(b) ($\Rightarrow$): Suppose f is measurable. Then

$$\{x : g(x) > \alpha\} = \begin{cases} \{x \in D : f(x) > \alpha\} & \text{if } \alpha \geq 0, \\ \{x \in D : f(x) > \alpha\} \cup D^c & \text{if } \alpha < 0, \end{cases}$$

which is measurable in either cases. Hence g is measurable.

($\Leftarrow$): The converse follows immediately from (a) since D is measurable and $f = g\big|_D$.

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5.* (3rd: P.71, Q22)

(a) Let f be an extended real-valued function with measurable domain D, and let $D_1 = \{x : f(x) = \infty\}$, $D_2 = \{x : f(x) = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable.

(b) Prove that the product of two measurable extended real-valued function is measurable. (Hint: unlike the case of sums, $f(x)g(x)$ is always of no ambiguity even when $f(x)$ and $g(x)$ are infinite.)

(c) If f and g are measurable extended real-valued functions and α is a fixed number, then $f + g$ is measurable if we define $f + g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$.

(d) Let f and g be measurable extended real-valued functions that are finite almost everywhere. Then $f + g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.

Solution. (a) ($\Rightarrow$): Suppose f is measurable. Then D_1 and D_2 are measurable as usual. Hence $D \setminus (D_1 \cup D_2)$ is measurable, and so is $f\big|_{D\setminus(D_1\cup D_2)}$ by 4(a).

($\Leftarrow$): Suppose D_1 and D_2 are measurable and $f\big|_{D\setminus(D_1\cup D_2)}$ is measurable. Then, for $\alpha \in \mathbb{R}$,

$$\{x : f(x) > \alpha\} = D_1 \cup \{x : f\big|_{D\setminus(D_1\cup D_2)} > \alpha\}$$

which is measurable. Thus f is measurable.

(b) Let $D_1 = \{fg = \infty\}$ and $D_2 = \{fg = -\infty\}$. Then

$$D_1 = \{f = \infty, g > 0\} \cup \{f = -\infty, g < 0\} \cup \{f > 0, g = \infty\} \cup \{f < 0, g = -\infty\},$$

hence is measurable. Similarly D_2 is also measurable. By (a), it suffices to show that $h := fg\big|_{D\setminus(D_1\cup D_2)}$ is measurable. Let $\alpha \in \mathbb{R}$. If $\alpha \geq 0$, then

$$\{x : h(x) > \alpha\} = \{x : f\big|_{D\setminus\{f=\pm\infty\}} \cdot g\big|_{D\setminus\{g=\pm\infty\}} > \alpha\},$$

which is measurable; if $\alpha < 0$, then

$$\{x : h(x) > \alpha\} = \{x : f(x) = 0\} \cup \{x : g(x) = 0\} \cup \{x : f\big|_{D\setminus\{f=\pm\infty\}} \cdot g\big|_{D\setminus\{g=\pm\infty\}} > \alpha\},$$

which is also measurable. Thus h is measurable.

Therefore fg is measurable.

(c) Let $D_1 := \{f + g = \infty\}$ and $D_2 := \{f + g = -\infty\}$. Then

$$D_1 = \{f \in \mathbb{R}, g = \infty\} \cup \{f = g = \infty\} \cup \{f = \infty, g \in \mathbb{R}\}$$

is measurable, and so is D_2. By (a), it suffices to show that $h := (f+g)\big|_{D\setminus(D_1\cup D_2)}$ is measurable. Let $\beta \in \mathbb{R}$. If $\beta \geq \alpha$, then

$$\{x : h(x) > \beta\} = \{x : f\big|_{D\setminus\{f=\pm\infty\}} + g\big|_{D\setminus\{g=\pm\infty\}} > \beta\},$$

which is measurable; if $\beta < \alpha$, then

$$\begin{aligned}\{x : h(x) > \beta\} = &\{f = \infty, g = -\infty\} \cup \{f = -\infty, g = +\infty\} \\ &\cup \{x : f\big|_{D\setminus\{f=\pm\infty\}} + g\big|_{D\setminus\{g=\pm\infty\}} > \beta\},\end{aligned}$$

which is also measurable. Thus h is measurable.

Therefore $f + g$ is measurable.

(d) Let D_1, D_2 and h be defined as in (c). Then the sets D_1, D_2, $\{x : h(x) > \beta\}$ can be written as unions of sets as in (c), possibly with an additional set of measure zero. Thus these sets are measurable and $f + g$ is measurable.

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