MATH 4050 Real Analysis

Suggested Solution of Homework 2

In this assignment, $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers. E is a subset of $\mathbb{R}$.

Recall that the limit superior of $\{x_n\}$ is defined by

$$\limsup x_n := \inf_n \sup_{k\geq n} x_k.$$

Clearly $z_n := \sup_{k\geq n} x_k$ is monotone decreasing, and hence

$$\lim_n z_n = \inf_n z_n = \limsup x_n, \tag{1}$$

where the limit is taken in the extended real number. Similarly the limit inferior of $\{x_n\}$ is given by

$$\liminf x_n := \sup_n \inf_{k\geq n} x_k = \lim_n \inf_{k\geq n} x_k. \tag{2}$$

1.* (3rd: P.39, Q12)

Show that $x = \lim x_n$ if and only if every subsequence of $\{x_n\}$ has in turn a subsequence that converges to x. How about $x \in \{-\infty, \infty\}$?

Solution. ($\Longrightarrow$) Suppose $\lim x_n = x$. Then every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x. Therefore $\{x_{n_k}\}$ has itself as a further subsequence that converges to x.

($\Longleftarrow$) Suppose on the contrary that $\{x_n\}$ does not converge to x. Then there exists $\varepsilon_0 > 0$ such that for all $N \in \mathbb{N}$, there is $n > N$ such that

$$|x_n - x| \geq \varepsilon_0.$$

Take $N = 1$, then we can find $n_1 > 1$ such that $|x_{n_1} - x| \geq \varepsilon_0$. Take $N = n_1$, we can find $n_2 > n_1$ such that $|x_{n_2} - x| \geq \varepsilon_0$. Continue in this way, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Now $\{x_{n_k}\}$ has no further subsequence that converges to x.

Similar results hold if $x = -\infty$ or ∞. ◀

2. (3rd: P.39, Q13)

Show that the real number l is the limit superior of the sequence $\{x_n\}$ if and only if (i) given $\varepsilon > 0$, $\exists\, n$ such that $x_k < l + \varepsilon$ for all $k \geq n$, and (ii) given $\varepsilon > 0$ and n, $\exists\, k \geq n$ such that $x_k > l - \varepsilon$.

Solution. We show that

(a) $\limsup x_n \leq l$ if and only if (i) holds ; and

(b) $\limsup x_n \geq l$ if and only if (ii) holds.

(a): By the definition of supremum and infinmum,

$$\limsup x_n \leq l \implies (\forall\, \varepsilon > 0)(\limsup x_n < l + \varepsilon) \implies (\forall\, \varepsilon > 0)(\inf_n \sup_{k \geq n} x_k < l + \varepsilon)$$

$$\implies (\forall\, \varepsilon > 0)(\exists\, n)(\sup_{k \geq n} x_k < l + \varepsilon) \implies (\forall\, \varepsilon > 0)(\exists\, n)(\forall\, k \geq n)(x_k < l + \varepsilon);$$

while on the other hand,

$$(\forall\, \varepsilon > 0)(\exists\, n)(\forall\, k \geq n)(x_k < l + \varepsilon) \implies (\forall\, \varepsilon > 0)(\exists\, n)(\sup_{k \geq n} x_k \leq l + \varepsilon)$$

$$\implies (\forall\, \varepsilon > 0)(\inf_n \sup_{k \geq n} x_k \leq l + \varepsilon) \implies (\forall\, \varepsilon > 0)(\limsup x_k \leq l + \varepsilon) \implies \limsup x_n \leq l.$$

(b): Similarly,

$$\limsup x_n \geq l \implies (\forall\, \varepsilon > 0)(\limsup x_n > l - \varepsilon) \implies (\forall\, \varepsilon > 0)(\inf_n \sup_{k \geq n} x_k > l - \varepsilon)$$

$$\implies (\forall\, \varepsilon > 0)(\forall\, n)(\sup_{k \geq n} x_k > l - \varepsilon) \implies (\forall\, \varepsilon > 0)(\forall\, n)(\exists\, k \geq n)(x_k > l - \varepsilon);$$

while on the other hand,

$$(\forall\, \varepsilon > 0)(\forall\, n)(\exists\, k \geq n)(x_k > l - \varepsilon) \implies (\forall\, \varepsilon > 0)(\forall\, n)(\sup_{k \geq n} x_k > l - \varepsilon)$$

$$\implies (\forall\, \varepsilon > 0)(\inf_n \sup_{k \geq n} x_k \geq l - \varepsilon) \implies (\forall\, \varepsilon > 0)(\limsup x_n \geq l - \varepsilon) \implies \limsup x_n \geq l.$$

Now the desired statement follows from (a) and (b) immediately.

Similarly, one can show that

(c) $\liminf x_n \geq l$ if and only if $\forall\, \varepsilon > 0$, $\exists\, n$ such that $x_k > l - \varepsilon$ for all $k \geq n$; and

(c) $\liminf x_n \leq l$ if and only if $\forall \varepsilon > 0$, $\forall\, n$, $\exists\, k \geq n$ such that $x_k < l + \varepsilon$.

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3.* (3rd: P.39, Q14)

Show that $\limsup x_n = \infty$ if and only if given Δ and n, $\exists\, k \geq n$ such that $x_k > \Delta$.

Solution. The statement follows immediately from (b) in question 2 and the fact that $x = \infty$ if and only if $x > \Delta$ for any $\Delta \in \mathbb{R}$. Indeed,

$$\limsup x_n = \infty \implies (\forall\, \Delta \in \mathbb{R})(\limsup x_n > \Delta) \implies (\forall\, \Delta \in \mathbb{R})(\forall\, n \in \mathbb{N})(\sup_{k \geq n} x_k > \Delta)$$

$$\implies (\forall\, \Delta \in \mathbb{R})(\forall\, n \in \mathbb{N})(\exists\, k \geq n)(x_k > \Delta).$$

while on the other hand,

$$(\forall\, \Delta \in \mathbb{R})(\forall\, n \in \mathbb{N})(\exists\, k \geq n)(x_k > \Delta) \implies (\forall\, \Delta \in \mathbb{R})(\forall\, n \in \mathbb{N})(\sup_{k \geq n} x_k > \Delta)$$

$$\implies (\forall\, \Delta \in \mathbb{R})(\limsup x_n \geq \Delta) \implies \limsup x_n = \infty.$$

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4. (3rd: P.39, Q15)

Show that $\liminf x_n \leq \limsup x_n$ and $\liminf x_n = \limsup x_n = l$ if and only if $l = \lim x_n$.

Solution. Clearly

$$\inf_{k\geq n} x_k \leq x_n \leq \sup_{k\geq n} x_k \qquad \text{for all } n \geq 1. \tag{3}$$

Hence, by (1) and (2), and letting $n \to \infty$, we have

$$\liminf x_n = \lim_n \inf_{k\geq n} x_k \leq \lim_n \sup_{k\geq n} x_k = \limsup x_n.$$

Suppose $\liminf x_n = \limsup x_n = l$. Then it follows from (3) and the Squeeze Theorem that $\lim x_n = l$.

Conversely, if $l = \lim x_n$, then for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $l - \varepsilon < x_k < l + \varepsilon$ for all $k \geq n$, so that

$$l - \varepsilon \leq \inf_{k\geq n} x_k \leq x_k \leq \sup_{k\geq n} x_k \leq l + \varepsilon \quad \text{for all } k \geq n.$$

Letting $n \to \infty$, we have $l - \varepsilon \leq \liminf x_n \leq \limsup x_n \leq l + \varepsilon$. As ε is arbitrary, we have $\liminf x_n = \limsup x_n = l$.

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5.* (3rd: P.39, Q16)

Prove that

$$\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n,$$

provided the right and left sides are not of the form $\infty - \infty$.

Solution. For all $n \geq 1$,

$$x_k + \inf_{j\geq n} y_j \leq x_k + y_k \quad \text{whenever } k \geq n,$$

so that

$$\sup_{k\geq n} x_k + \inf_{j\geq n} y_j \leq \sup_{k\geq n}(x_k + y_k).$$

By (1) and (2), we can let $n \to \infty$ on both sides and obtain

$$\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n),$$

provided the left side is not of the form $\infty - \infty$.

On the other hand, for all $n \geq 1$,

$$x_j + y_j \leq \sup_{k\geq n} x_k + \sup_{k\geq n} y_k \quad \text{whenever } j \geq n,$$

so that

$$\sup_{k\geq n}(x_k+y_k) \leq \sup_{k\geq n} x_k + \sup_{k\geq n} y_k.$$

Again letting $n\to\infty$, we obtain

$$\limsup(x_n+y_n) \leq \limsup x_n + \limsup y_n,$$

provided the right side is not of the form $\infty - \infty$.

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6. (3rd: P.39, Q17)

Prove that if $x_n > 0$ and $y_n \geq 0$, then

$$\limsup(x_n y_n) \leq (\limsup x_n)(\limsup y_n),$$

provided the product on the right is not of the form $0\cdot\infty$.

Solution. For all $n \geq 1$,

$$0 \leq x_k \leq \sup_{j\geq n} x_j \quad \text{and} \quad 0 \leq y_k \leq \sup_{j\geq n} y_j \quad \text{whenever } k \geq n,$$

so that

$$0 \leq x_k y_k \leq (\sup_{j\geq n} x_j)(\sup_{j\geq n} y_j) \quad \text{whenever } k \geq n.$$

Thus, for all $n \geq 1$,

$$\sup_{k\geq n}(x_k y_k) \leq (\sup_{k\geq n} x_k)(\sup_{k\geq n} y_k).$$

Using (1) and (2), and letting $n\to\infty$, we have

$$\limsup(x_n y_n) \leq (\limsup x_n)(\limsup y_n),$$

provided the right side is not of the form $0\cdot\infty$. ◀

7. (3rd: P.46, Q27)

Recall that $x\in\mathbb{R}$ is called a *point of closure* of E if each neighbourhood of x intersects E. Show that x is a point of closure of E if and only if there is a sequence $\{y_n\}$ with $y_n \in E$ and $x = \lim y_n$.

Solution. Suppose x is a point of closure of E. Then the open ball $B(x, 1/n)$, which is centred at x and of radius $1/n$, intersects E for all $n \geq 1$. Pick $y_n \in E \cap B(x, 1/n)$ for each n. Then $\{y_n\}$ is a sequence in E such that $\lim y_n = x$, since $|y_n - x| < 1/n$ for all n.

On the other hand, suppose $\{y_n\}$ is a sequence in E such that $x = \lim y_n$. Let U be a neighbourhood of x. Then $y_n \to x$ implies that $y_n \in U$ for all sufficiently large n. In particular, $U \cap E \neq \emptyset$. ◀

8. (3rd: P.46, Q28; 4th: P.20, Q30(i))

   A number x is called an *accumulation point* of a set E if it is a point of closure of $E \setminus \{x\}$. Show that the set E' of accumulation points of E is a closed set.

   Solution. We would like to show that the complement of E' is open. Let $x \in (E')^c$. Then x is not a point of closure of $E \setminus \{x\}$. Hence, by definition, there is an open neighbourhood U of x such that $U \cap (E \setminus \{x\}) = \emptyset$. We claim that every $y \in U$ is not an accumulation point of E, so that $x \in U \subseteq (E')^c$, and hence $(E')^c$ is open.

   Let $y \in U \setminus \{x\}$. Since $U \setminus \{x\}$ is open, there is a neighbourhood V of y such that $V \subseteq U \setminus \{x\}$. Hence

   $$V \cap (E \setminus \{y\}) \subseteq (U \setminus \{x\}) \cap E = \emptyset.$$

   Thus y is not a point of closure of $E \setminus \{y\}$, that is, y is not an accumulation point of E.

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9. (3rd: P.46, Q29; 4th: P.20, Q30(ii))

   Show that $\overline{E} = E \cup E'$.

   Solution. Recall that $\overline{E}$ is the set of all point of closure of E. From the definitions, it is clear that $E \cup E' \subseteq \overline{E}$. On the other hand, if $x \in \overline{E} \setminus E$, then for every neighbourhood U of x,

   $$U \cap (E \setminus \{x\}) = U \cap E \neq \emptyset.$$

   Hence $x \in E'$. Therefore $\overline{E} \subseteq E \cup E'$. ◀

10. (3rd: P.46, Q30; 4th: P.20, Q31)

    A set E is called *isolated* if $E \cap E' = \emptyset$. Show that every isolated set of real numbers is countable.

    Solution. Suppose E is isolated. Then no point in E is an accumulation point of E, whence, for all $x \in E$, there is $r_x > 0$ such that $(x - r_x, x + r_x) \cap (E \setminus \{x\}) = \emptyset$. Let $I_x = (x - r_x/2, x + r_x/2)$. Then $\{I_x : x \in E\}$ is a collection of open intervals such that

    $$I_x \cap I_y = \emptyset \quad \text{if } x, y \in E,\ x \neq y.$$

    For otherwise, $u \in I_x \cap I_y \implies |x-y| \leq |x-u| + |u-y| < r_x/2 + r_y/2 \leq \max\{r_x, r_y\}$, contradicting $x \notin I_y$ and $y \notin I_x$.

    By the density of $\mathbb{Q}$, for every $x \in E$, we can find $\varphi(x) \in \mathbb{Q}$ such that $\varphi(x) \in I_x$. Now $\varphi : E \to \mathbb{Q}$ is an injection since $\{I_x : x \in E\}$ are pairwise disjoint. Therefore E is countable.

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11.* Let $f : [0,1] \to [m, M]$ with Riemann upper integral $\alpha = (\mathcal{R}) \overline{\int_0^1} f(x)\,dx$. Show there is a sequence (ψ_n) of step-functions such that $\int_0^1 \psi_n(x)\,dx \to \alpha$ and

$$\psi_n(x) \downarrow \bar{f}(x) \qquad \forall x \in X := [0,1] \setminus \{k/2^n : n \in \mathbb{N},\ k = 0, 1, \ldots, 2^n\},$$

where

$$\bar{f}(x) := \inf\{f^\delta(x) : \delta > 0\}, \quad \forall x \in [0,1]$$

with each

$$f^\delta(x) := \sup\{f(u) : u \in V_\delta(x) \cap [0,1]\}, \quad \forall x \in [0,1].$$

Solution. Let P_n be the partition that divides $[0,1]$ into 2^n-many subintervals of equal length $1/2^n$. Define a step-function ψ_n by

$$\psi_n(x) := \sum_{k=1}^{2^n} \sup\{f(x) : \frac{k-1}{2^n} < x \le \frac{k}{2^n}\} \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n}]}.$$

Then clearly $\psi_{n+1}(x) \le \psi_n(x)$ for all $x \in X$.

Since $\|P_n\| \to 0$ as $n \to \infty$, we have

$$\int_0^1 \psi_n(x)\,dx = U(f, P_n) \to (\mathcal{R}) \overline{\int_0^1} f(x)\,dx = \alpha,$$

where $U(f, P_n)$ is the upper sum of f with respect to the partition P_n.

Let $x_0 \in X$ and suppose x_0 lies in I, one of the above subintervals. Then x_0 must be in the interior of I, so there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq I$. Then $f^\delta(x_0) \le \psi_n(x_0)$, and hence $\bar{f}(x_0) \le \psi_n(x_0)$ for each n.

Conversely, let $x_0 \in X$ and $\delta > 0$. Take $n \in \mathbb{N}$ such that $1/2^n < \delta$. Now if I is one of the above subintervals that contain x_0, then $x_0 \in I \subseteq (x_0 - \delta, x_0 + \delta)$ as the length of I is smaller than δ. Thus $\psi_n(x_0) \le f^\delta(x_0)$, so that

$$\inf\{\psi_n(x_0) : n \in \mathbb{N}\} \le \psi_n(x_0) \le f^\delta(x_0).$$

As $\delta > 0$ is arbitrary, we have $\inf\{\psi_n(x_0) : n \in \mathbb{N}\} \le \bar{f}(x_0)$. Hence

$$\lim \psi_n(x_0) = \inf\{\psi_n(x_0) : n \in \mathbb{N}\} = \bar{f}(x_0).$$

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