MATH4050 Real Analysis
Assignment 5

There are 9 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

1. (3rd: P.70, Q19)
   Let D be a dense set of real numbers, that is, a set of real numbers such that every interval contains an element of D. Let f be an extended real-valued function on $\mathbb{R}$ such that $\{x : f(x) > \alpha\}$ is measurable for each $\alpha \in D$. Show that f is measurable.

2. (3rd: P.70, Q20; 4th: P.63, Q19 and P.64 Q20)
   Show that the sum and product of two simple functions are simple. Show that for any $A, B \subset \mathbb{R}$,
   $$\begin{aligned}\chi_{A\cap B} &= \chi_A \cdot \chi_B\\ \chi_{A\cup B} &= \chi_A + \chi_B - \chi_{A\cap B}\\ \chi_{\widetilde{A}} &= 1 - \chi_A.\end{aligned}$$
   (Note: $\widetilde{A} =$ complement of A)

3. (3rd: P.71, Q23)
   Prove Proposition 22 (3rd ed.) by establishing the following lemmas:

   a. Given a measurable function f on $[a,b]$ that takes the values $\pm\infty$ only on a set of measure zero, and given $\varepsilon > 0$, there is an M such that $|f| \le M$ except on a set of measure less than $\dfrac{\varepsilon}{3}$.
   b. Let f be a measurable function on $[a,b]$. Given $\varepsilon > 0$ and M, there is a simple function φ such that $|f(x) - \varphi(x)| < \varepsilon$ except where $|f(x)| \ge M$. If $m \le f \le M$, then we may take φ so that $m \le \varphi \le M$.
   c. Given a simple function φ on $[a,b]$, there is a step function g on $[a,b]$ such that $g(x) = \varphi(x)$ except on a set of measure less than $\dfrac{\varepsilon}{3}$. [Hint: Use Proposition 15 (3rd ed.).] If $m \le \varphi \le M$, then we can take g so that $m \le g \le M$.
   d. Given a step function g on $[a,b]$, there is a continuous function h such that $g(x) = h(x)$ except on a set of measure less than $\dfrac{\varepsilon}{3}$. If $m \le g \le M$, then we may take h so that $m \le h \le M$.

   Proposition 15 is the Littlewood's first principle (See lecture notes Ch3 P.12-13).

   Proposition 22: Let f be a measurable function defined on an interval $[a,b]$, and assume that f takes the value $\pm\infty$ only on a set of measure zero. Then given $\varepsilon > 0$, we can find a step function g and a continuous function h such that
   $$|f - g| < \varepsilon \text{ and } |f - h| < \varepsilon$$
   except on a set of measure less than ε; i.e. $m(\{x : |f(x)-g(x)| \ge \varepsilon\}) < \varepsilon$ and $m(\{x : |f(x)-h(x)| \ge \varepsilon\}) < \varepsilon$. If in addition $m \le f \le M$, then we may choose the functions g and h such that $m \le g \le M$ and $m \le h \le M$.

4. *(3rd: P.71, Q24; 4th: P.59, Q7)
   Let f be measurable and B a Borel set. Show that $f^{-1}[B]$ is a measurable set. [Hint: The class of sets for which $f^{-1}[E]$ is measurable is a σ-algebra.]

5. *(3rd: P.71, Q25; 4th: P.59, Q10)
   Show that if f is a measurable real-valued function and g a continuous function defined on $(-\infty, \infty)$, then $g \circ f$ is measurable.

6. *(3rd: P.73, Q29)
   Given an example to show that we must require $m(E) < \infty$ in Proposition 23 (3rd ed.).

   Proposition 23 is the claim $(*)$ in the proof of Egoroff's theorem in the lecture notes (Ch3, P.25), except the pointwise convergence a.e. on E is replaced by pointwise convergence on E.

7. (3rd: P.73, Q30)
   Prove Egoroff's Theorem.

8. *(3rd: P.74, Q31)
   Prove Lusin's Theorem: Let f be a measurable real-valued function on an interval $[a, b]$. Then given $\delta > 0$, there is a continuous function φ on $[a, b]$ such that $m(x : f(x) \neq \varphi(x)) < \delta$. Can you do the same on the interval $(-\infty, \infty)$?

9. (3rd: P.74, Q32)
   Show that Proposition 23 (3rd ed.) (See Question 6) need not be true if the integer variable n is replaced by a real variable t; that is, construct a family $\{f_t\}$ of measurable real-valued functions on $[0, 1]$ such that for each x we have $\lim_{t \to 0} f_t(x) = 0$, but for some $\delta > 0$ we have $m^*(\{x : f_t(x) > \frac{1}{2}\}) > \delta$.