MATH4050 Real Analysis
Assignment 4

There are 5 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

1. (3rd: P.64, Q9)
Show that if E is a measurable set, then each translate $E + y$ of E is also measurable.

2. (3rd: P.64, Q10; 4th: P.43, Q24)
Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

3. (3rd: P.64, Q11; 4th: P.43, Q25)
Show that the condition $m(E_1) < \infty$ is necessary in Proposition 14 (3rd ed.) by giving a decreasing sequence $\{E_n\}$ of measurable sets with $\phi = \bigcap E_n$ and $m(E_n) = \infty$ for each n.

Proposition 14: Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets, that is, a sequence with $E_{n+1} \subset E_n$ for each n. Let $m(E_1)$ be finite. Then

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n).$$

4. (3rd: P.70, Q21; 4th: P.59, Q2,6)
   a. Let D and E be measurable sets and f a function with domain $D \cup E$. Show that f is measurable if and only if its restrictions to D and E are measurable.
   b. Let f be a function with measurable domain D. Show that f is measurable iff the function g defined (on $\mathbb{R}$) by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

5. (3rd: P.71, Q22)
   a. Let f be an extended real-valued function with measurable domain D, and let $D_1 = \{x : f(x) = \infty\}$, $D_2 = \{x : f(x) = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable.
   b. Prove that the product of two measurable extended real-valued functions is measurable (Hint: unlike the case of sums, $f(x)g(x)$ is always of no ambiguity even when $f(x)$ and $g(x)$ are infinite.)
   c. If f and g are measurable extended real-valued functions and α a fixed number, then $f + g$ is measurable if we define $f + g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$.
   d. Let f and g be measurable extended real-valued functions that are finite almost everywhere. Then $f + g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.