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FULLY HÖLDERIAN STABLE MINIMUM WITH RESPECT TO BOTH TILT AND PARAMETER PERTURBATIONS*

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Abstract. When the objective function undergoes both a tilt perturbation and a general parameter perturbation, this paper considers the notions of a fully stable Hölder minimizer, a uniform Hölder growth condition, and a fully stable (q, s)-minimum, where the last notion reduces to the tilt-stable minimum by Levy, Poliquin, and Rockafellar [*SIAM J. Optim.*, 10 (2000), pp. 580–604] and the fully Hölder stable minimum by Mordukhovich and Nghia [*SIAM J. Optim.*, 24 (2014), pp. 1344–1381] as special cases by taking $(q, s) = (2, 2)$ and $(q, s) = (2, 1)$, respectively. Under weak-$(\mathcal{BCQ})$ (a new constraint qualification), by using the techniques of variational analysis, we establish relationships among these notions and provide several characterizations for fully stable (q, s)-minima, which improve and generalize some existing results in the recent literature.

Key words. fully Hölderian stable minimum, uniform Hölder growth condition, metric regularity

AMS subject classifications. 90C31, 90C30, 49J52

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1. Introduction.

Given a proper lower semicontinuous function $f : X \times P \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ with X being a Banach space and P a metric parameter space, consider the following perturbed optimization problem with two parameters:

$$\mathcal{P}(x^*, p) \qquad \min f(x, p) - \langle x^*, x\rangle \text{ over } x \in X,$$

where x^* is in the dual space X^*. In 2000, Levy, Poliquin, and Rockafellar [5] first studied the full Lipschitz stability of locally optimal solutions to $\mathcal{P}(x^*, p)$ in the following sense: *given* $(\bar{x}^*, \bar{p}) \in X^* \times P$, *a point* $\bar{x} \in X$ *is said to be a fully Lipschitz-stable solution to* $\mathcal{P}(x^*, p)$ *at* $(\bar{x}^*, \bar{p})$ *if there exist* $r, L \in (0, +\infty)$, *a neighborhood* $W \subset X^* \times P$ *of* $(\bar{x}^*, \bar{p})$, *and a mapping* M_r *from* W *to the open ball* $B_X(\bar{x}, r)$ *with center* $\bar{x}$ *and radius* r *such that* $M_r(\bar{x}^*, \bar{p}) = \{\bar{x}\}$,

$$\underset{x\in B_X(\bar{x},r)}{\arg\min} \{f(x, p) - \langle x^*, x\rangle\} = \{M_r(x^*, p)\} \quad \forall (x^*, p) \in W, \tag{1.1}$$

and

$$\|M_r(x_1^*, p_1) - M_r(x_2^*, p_2)\| \le L(\|x_1^* - x_2^*\| + d(p_1, p_2)) \quad \forall (x_1^*, p_1), (x_2^*, p_2) \in W. \tag{1.2}$$

They provided second-order conditions for $\bar{x}$ to be a fully Lipschitz-stable solution to $\mathcal{P}(x^*, p)$ at $(\bar{x}^*, \bar{p})$ under the assumption that $f : X \times P \to \overline{\mathbb{R}}$ is parametrically

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subdifferentially continuous and prox-regular at $(\bar{x}, \bar{p}, \bar{x}^*)$ and satisfies the following basic constraint qualification: *there exist* $L \in (0,\ +\infty)$ *and neighborhoods* $U \subset P$ *of* $\bar{p}$ *and* $G \subset X \times \mathbb{R}$ *of* $(\bar{x}, f(\bar{x}, \bar{p}))$ *such that*

$$(\mathcal{BCQ}) \qquad \operatorname{epi}(f_{p_1}) \cap G \subset \operatorname{epi}(f_{p_2}) + Ld(p_1, p_2)(B_X \times [-1,\ 1]) \quad \forall p_1, p_2 \in U,$$

where $\operatorname{epi}(f_p)$ *denotes the epigraph of* $f_p := f(\cdot, p)$ *and* B_X *denotes the closed unit ball of* X. They also studied the Lipschitz continuity of the following infimum function:

$$(u^*, p) \mapsto m_r(u^*, p) := \inf_{x \in B_X(\bar{x}, r)} \{f(x, p) - \langle u^*, x\rangle\}. \tag{1.3}$$

In an earlier paper than [5], Poliquin and Rockafellar [15] studied the Lipschitz-stable solutions known as tilt-stable minimizers for the case in which $P = \{\bar{p}\}$. Recently, Zheng and Ng [17] further studied tilt-stable q-order local minimizers with $1 \leq q < +\infty$. For the case in which P is a general metric space, with $d(p_1, p_2)^{1/2}$ replacing $d(p_1, p_2)$ in (1.2), Mordukhovich and Nghia [8, 10, 11] considered the so-called full Hölderian stability of the solutions to $\mathcal{P}(x^*, p)$: $\bar{x} \in X$ *is said to be a fully Hölder-stable solution to* $\mathcal{P}(x^*, p)$ *at* $(\bar{x}^*, \bar{p})$ *if there exist* $r, L \in (0,\ +\infty)$, *a neighborhood* W *of* $(\bar{x}^*, \bar{p})$, *and a mapping* $M_r : W \to B_X(\bar{x}, r)$ *with* $M_r(\bar{x}^*, \bar{p}) = \{\bar{x}\}$ *such that* (1.1) *holds and*

$$\|M_r(x_1^*, p_1) - M_r(x_2^*, p_2)\| \leq L(\|x_1^* - x_2^*\| + d(p_1, p_2)^{\frac{1}{2}}) \quad \forall (x_1^*, p_1), (x_2^*, p_2) \in W. \tag{1.4}$$

This property is of course weaker than the full Lipschitz-stability because $d(p_1, p_2) \leq d(p_1, p_2)^{1/2}$ for all $p_1, p_2 \in P$ close to $\bar{p}$. Mordukhovich and Nghia [8, 10, 11] provided some characterizations of full Hölder-stability under the assumptions that $(\mathcal{BCQ})$ is satisfied and f is parametrically subdifferentially continuous and prox-regular at $(\bar{x}, \bar{p}, \bar{x}^*)$. Replacing $\|M_r(x_1^*, p_1) - M_r(x_2^*, p_2)\|$ in (1.4) by $\|M_r(x_1^*, p_1) - M_r(x_2^*, p_2)\|^{q_1}$ and $d(p_1, p_2)^{1/2}$ by $d(p_1, p_2)^{q_2}$, it is natural to consider the following more general stability notion: *given* $q_1, q_2 \in (0, +\infty)$ *and* $(\bar{x}, \bar{p}, \bar{x}^*) \in X \times P \times X^*$, *there exist* $r, L \in (0,\ +\infty)$, *a neighborhood* W *of* $(\bar{x}^*, \bar{p})$, *and a mapping* $M_r : W \to B_X(\bar{x}, r)$ *with* $M_r(\bar{x}^*, \bar{p}) = \bar{x}$ *such that* (1.1) *holds and*

$$\|M_r(x_1^*, p_1) - M_r(x_2^*, p_2)\|^{q_1} \leq L(\|x_1^* - x_2^*\| + d(p_1, p_2)^{q_2}) \quad \forall (x_1^*, p_1), (x_2^*, p_2) \in W. \tag{1.5}$$

In this paper, we consider the following notion: *given* $q, s \in (0,\ +\infty)$ *and* $(\bar{x}, \bar{p}, \bar{x}^*) \in X \times P \times X^*$, *one may say that* $\bar{x}$ *is a fully stable* (q, s)*-solution to* $\mathcal{P}(x^*, p)$ *at* $(\bar{x}^*, \bar{p})$ *if there exist* $r, l \in (0,\ +\infty)$, *a neighborhood* W *of* $(\bar{x}^*, \bar{p})$, *and a mapping* $M_r : W \to B_X(\bar{x}, r)$ *with* $M_r(\bar{x}^*, \bar{p}) = \bar{x}$ *such that* (1.1) *holds and*

$$\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\|^q \leq l(\|u_1^* - u_2^*\| \|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\| + d(p_1, p_2)^s) \tag{1.6}$$

for all $(u_1^*, p_1), (u_2^*, p_2) \in W$. *In this case, we also say that* f *gives a fully stable* (q, s)*-minimum at* $\bar{x}$ *for* $(\bar{x}^*, \bar{p})$.

Since $\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\| \leq 2r$ for all $(u_1^*, p_1), (u_2^*, p_2) \in W$, (1.6) implies clearly that inequality (1.5) holds with $(q_1, q_2) = (q, s)$ and $L = l(2r + 1)$. We shall also prove that if $q > 1$ and $(q_1, q_2) = (q - 1, s)$ then inequality (1.5) implies (1.6). For the case in which $q = 2$, we shall prove that (1.6) holds for some $l \in (0,\ +\infty)$ if and only if (1.5) holds for some $L \in (0,\ +\infty)$ with $(q_1, q_2) = (q, \frac{s}{2})$ (see Proposition 4.1). Thus our notion defined by (1.6) in the special case in which $(q, s) = (2, 2)$ reduces to the corresponding notion by Levy, Poliquin, and Rockafellar [5], while in another special case in which $(q, s) = (2, 1)$ it reduces to the one by Mordukhovich and Nghia [8]. Moreover, when the objective function undergoes both a tilt perturbation and a parameter perturbation, we introduce the notions of a fully stable Hölder minimizer, a uniform Hölder growth condition, and an S-condition, and obtain their interrelated results, new even in the special case in which the parameter space P is a singleton.

The rest of the paper is organized as follows. Section 2 presents some basic definitions and properties of variational analysis used in the main body of the paper. In section 3, we introduce and study notions of the fully Hölder stable minimizer, uniform Hölder growth condition, and S-condition. In this connection, we adopt a new constraint qualification (weak-$(\mathcal{BCQ})$) which is weaker than $(\mathcal{BCQ})$ and plays a role somewhat similar to that played by $(\mathcal{BCQ})$ in the work by Levy, Poliquin, and Rockafellar [5] and Mordukhovich and Nghia [8, 10, 11]. In section 4, in terms of the subdifferential of the concerned function, we give characterizations for a fully stable (q, s)-minimum and a fully stable q_1-order minimizer. Our results are new even in the special case in which $(q, s) = (2, 1)$ or $(q, s) = (2, 2)$.

2. Preliminaries. In the remainder of this paper, unless otherwise specified, X is assumed to be an Asplund space, i.e., X is a Banach space such that its every separable subspace has a separable dual. This is a broad class of spaces including all reflexive spaces; see [6, 16] for more details and references. For $\bar{x} \in X$ and $\delta > 0$, let $B_X(\bar{x}, \delta)$ and $B_X[\bar{x}, \delta]$ denote the open ball and closed ball centered at $\bar{x}$ with radius δ in X, respectively.

Given a proper lower semicontinuous function $\varphi : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, let $\mathrm{dom}(\varphi)$ and $\mathrm{epi}(\varphi)$ denote the domain and the epigraph of φ, respectively, that is,

$$\mathrm{dom}(\varphi) := \{x \in X : \varphi(x) < +\infty\} \quad \text{and} \quad \mathrm{epi}(\varphi) := \{(x, t) \in X \times \mathbb{R} : \varphi(x) \le t\}.$$

Recall that the Fréchet subdifferential of φ at $x \in \mathrm{dom}(\varphi)$ is defined as

$$\hat{\partial}\varphi(x) := \left\{ x^* \in X^* : \liminf_{y \to x} \frac{\varphi(y) - \varphi(x) - \langle x^*, y - x\rangle}{\|y - x\|} \ge 0 \right\},$$

and $\hat{\partial}\varphi(x)$ is understood as $\emptyset$ for $x \notin \mathrm{dom}(\varphi)$. Let $\partial\varphi(x)$ denote the Mordukhovich limiting subdifferential of φ at x, that is, $x^* \in \partial\varphi(x)$ if and only if there exist sequences $\{x_n\} \subset \mathrm{dom}(\varphi)$ and $\{x_n^*\} \subset X^*$ such that

$$(x_n, \varphi(x_n)) \to (x, \varphi(x)), \quad x_n^* \overset{w^*}{\to} x^*, \quad \text{and} \quad x_n^* \in \hat{\partial}\varphi(x_n) \quad \forall n \in \mathbb{N},$$

where $\overset{w^*}{\to}$ denotes the convergence with respect to the weak* topology of the dual X^*. In the case in which φ is convex, it is well known that

$$\hat{\partial}\varphi(x) = \partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \varphi(x + h) - \varphi(x) \text{ for all } h \in X\}.$$

The following rules for the subdifferentials (cf. [16, 6]) are useful to us.

LEMMA 2.1. *Let X be an Asplund space and let $f_1, f_2 : X \to \overline{\mathbb{R}}$ be proper lower semicontinuous functions such that f_2 is locally Lipschitz at $\bar{x} \in \mathrm{dom} f_1$. Then the following assertions hold:*

(i) $0 \in \hat{\partial} f_1(\bar{x}) \subset \partial f_1(\bar{x})$ *whenever $\bar{x}$ is a local minimizer of f_1.*

(ii) $\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x})$.

Let P be a metric space and $f : X \times P \to \overline{\mathbb{R}}$ be a proper function. Given a p in P, define $f_p : X \to \overline{\mathbb{R}}$ such that

$$f_p(x) := f(x, p) \quad \forall x \in X, \tag{2.1}$$

and let

$$\hat{\partial}_x f(x,p) := \hat{\partial} f_p(x) \quad \text{and} \quad \partial_x f(x,p) := \partial f_p(x). \tag{2.2}$$

To study the full Lipschitzian stability and full Hölderian stability for perturbed optimization problem $\mathcal{P}(x^*,p)$, both Levy, Poliquin, and Rockafellar [5] and Mordukhovich and Nghia [8, 10, 11] used the following continuous parametric prox-regularity (as their essential assumption): *f is said to be prox-regular in x at $\bar{x}$ for $\bar{x}^*$ with compatible parameterization by p at $\bar{p}$ if $\bar{x}^* \in \partial_x f(\bar{x},\bar{p})$ and there exist $\varepsilon, r \in (0, +\infty)$ and a neighborhood $G\times U\times V$ of $(\bar{x},\bar{p},\bar{x}^*)$ such that*

$$f(x',p) \geq f(x,p) + \langle x^*, x'-x\rangle - \frac{r}{2}\|x'-x\|^2 \quad \forall x' \in G \tag{2.3}$$

whenever $(x,p) \in G\times U$, $x^* \in \partial_x f(x,p)\cap V$, and $f(x,p) \leq f(\bar{x},\bar{p}) + \varepsilon$; f is said to be continuously prox-regular in x at $\bar{x}$ for $\bar{x}^*$ with compatible parameterization by p at $\bar{p}$ if, in addition, $f(x,p)$ is continuous as a function of $(x,p,x^*) \in \mathrm{gph}(\partial_x f)$ at $(\bar{x},\bar{p},\bar{x}^*)$.

It is easy to verify that a lower semicontinuous function $f : X\times P \to \overline{\mathbb{R}}$ is continuously prox-regular in x at $\bar{x}$ for $\bar{x}^*$ with compatible parameterization by p at $\bar{p}$ if and only if there exist $r \in (0, +\infty)$ and a neighborhood $G\times U\times V$ of $(\bar{x},\bar{p},\bar{x}^*)$ such that (2.3) holds whenever $(x,p) \in G\times U$ and $x^* \in \partial_x f(x,p)\cap V$.

As a generalization of the above continuous parametric prox-regularity, we adopt the following notion.

DEFINITION 2.1. *Let $q \in (1,+\infty)$ and $(\bar{x},\bar{p},\bar{x}^*) \in \mathrm{gph}(\partial_x f)$. The objective function f is said to be continuously q-regular in x at $\bar{x}$ for $(\bar{x}^*,\bar{p})$ if there exist $\rho \in (0, +\infty)$ and a neighborhood $G\times U\times V$ of $(\bar{x},\bar{p},\bar{x}^*)$ such that*

$$f(x',p) \geq f(x,p) + \langle x^*, x'-x\rangle - \rho\|x'-x\|^q \quad \forall x' \in G \tag{2.4}$$

whenever $(x,p) \in G\times U$ and $x^* \in \partial_x f(x,p)\cap V$.

For the special case in which $q = 2$, the continuous q-regularity reduces to the continuous prox-regularity of Levy, Poliquin, and Rockafellar [5].

Given two subsets A and B of the product $X\times X^*$ with $A\subset B$, recall that A is monotone if

$$0 \leq \langle x_2^* - x_1^*, x_2 - x_1\rangle \quad \forall (x_1,x_1^*),(x_2,x_2^*) \in A$$

and that A is a maximal monotone subset of B if A is monotone and there exists no other monotone subset of B containing A (cf. [15]). We say that a multifunction $T : X \rightrightarrows X^*$ is maximally monotone if $\mathrm{gph}(T)$ is a maximally monotone subset of $X\times X^*$. For a maximally monotone multifunction T, we don't know whether or not T is locally maximally monotone at $(\bar{x},\bar{x}^*) \in \mathrm{gph}(T)$ in the following sense: for any neighborhood U of $(\bar{x},\bar{x}^*)$ there exists a neighborhood V of $(\bar{x},\bar{x}^*)$ with $V\subset U$ such that $\mathrm{gph}(T)\cap V$ is a maximally monotone subset of V. However, we do have the following lemma, which is useful to us.

LEMMA 2.2. *Let r, σ be positive real numbers and $\delta \in (0, r\sigma]$. Suppose that X is a Hilbert space, $T : X \rightrightarrows X$ is a maximally monotone operator, and that $(\bar{x},\bar{x}^*) \in \mathrm{gph}(T)$. Then $\mathrm{gph}(T+\sigma I)\cap(B(\bar{x},r)\times B(\bar{x}^*+\sigma\bar{x},\delta))$ is a maximal monotone subset of $B(\bar{x},r)\times B(\bar{x}^*+\sigma\bar{x},\delta)$.*

Proof. Since T is maximally monotone, the mapping $x^* \mapsto (T+\sigma I)^{-1}(x^*)$ is single valued on X and

$$\|(T+\sigma I)^{-1}(v_1) - (T+\sigma I)^{-1}(v_2)\| \leq \sigma^{-1}\|v_1 - v_2\| \quad \forall v_1,v_2 \in X \tag{2.5}$$

(cf. [2, Corollary 23.10]). Then, noting that $(T+\sigma I)^{-1}(\bar{x}^*+\sigma\bar{x})=\bar{x}$, it follows from (2.5) that

$$(T+\sigma I)^{-1}(B(\bar{x}^*+\sigma\bar{x},\delta))\subset B(\bar{x},r). \tag{2.6}$$

Let (x_0,y_0) be an arbitrary element in $B(\bar{x},r)\times B(\bar{x}^*+\sigma\bar{x},\delta)$ such that

$$0\le\langle v-y_0,u-x_0\rangle\quad\forall(u,v)\in\operatorname{gph}(T+\sigma I)\cap(B(\bar{x},r)\times B(\bar{x}^*+\sigma\bar{x},\delta)). \tag{2.7}$$

To prove the lemma, we only need to show that $\operatorname{gph}(T+\sigma I)\cap(B(\bar{x},r)\times B(\bar{x}^*+\sigma\bar{x},\delta))$ is a maximal monotone subset of $B(\bar{x},r)\times B(\bar{x}^*+\sigma\bar{x},\delta)$. To do this, it suffices to show that $x_0=(T+\sigma I)^{-1}(y_0)$. To do this, let $h_0:=(T+\sigma I)^{-1}(y_0)-x_0$, and take a sequence $\{t_n\}\subset(0,+\infty)$ convergent to 0 such that $y_0-t_nh_0\in B(\bar{x}^*+\sigma\bar{x},\delta)$ for all $n\in\mathbb{N}$. Let $u_n:=(T+\sigma I)^{-1}(y_0-t_nh_0)$. Then, by (2.6), one has

$$(u_n,y_0-t_nh_0)\in\operatorname{gph}(T+\sigma I)\cap(B(\bar{x},r)\times B(\bar{x}^*+\sigma\bar{x},\delta)). \tag{2.8}$$

It follows from (2.7) that

$$0\le\langle -t_nh_0,u_n-x_0\rangle=\langle -t_nh_0,(T+\sigma I)^{-1}(y_0-t_nh_0)-(T+\sigma I)^{-1}(y_0)+h_0\rangle.$$

Hence

$$\begin{aligned}\|h_0\|^2&\le-\left\langle h_0,(T+\sigma I)^{-1}(y_0-t_nh_0)-(T+\sigma I)^{-1}(y_0)\right\rangle\\&\le\|h_0\|\|(T+\sigma I)^{-1}(y_0-t_nh_0)-(T+\sigma I)^{-1}(y_0)\|\\&\le\sigma^{-1}t_n\|h_0\|^2\end{aligned}$$

(the last inequality holds because of (2.5)). Since $t_n\to 0$, this implies that $h_0=0$, namely $x_0=(T+\sigma I)^{-1}(y_0)$. The proof is complete. ☐

The fact that r, σ, and δ in Lemma 2.2 are independent on T will play an important role in our analysis later.

3. Fully stable Hölder minimizers. Let X be a Banach space, $\varphi:X\to\overline{\mathbb{R}}$ be a proper lower semicontinuous function, and let $\bar{x}$ be a point in $\operatorname{dom}(\varphi)$. When φ undergoes small linear perturbations, many researchers (cf. [3, 4, 7, 9, 15]) studied the stable second-order (local) minimizer in the following sense: *$\bar{x}$ is said to be a stable second-order minimizer of φ if there exist $\delta,r,\kappa\in(0,\ +\infty)$ and a mapping $\vartheta:B_{X^*}(0,\delta)\to B_X(\bar{x},r)$ such that $\vartheta(0)=\bar{x}$ and*

$$\kappa\|x-\vartheta(u^*)\|^2\le\varphi(x)-\varphi(\vartheta(u^*))-\langle u^*,x-\vartheta(u^*)\rangle\quad\forall(x,u^*)\in B_X(\bar{x},r)\times B_{X^*}(0,\delta). \tag{3.1}$$

This notion was extended recently in [17, 18] to the so-called stable Hölder minimizer (namely, by replacing $\|x-\vartheta(u^*)\|^2$ in (3.1) by $\|x-\vartheta(u^*)\|^q$). In this section, we mainly consider the more general stable Hölder minimizer with "double parameterization variables" u^* and p. Throughout the remainder of this paper, let P be a metric space and $f:X\times P\to\overline{\mathbb{R}}$ be a proper lower semicontinuous function. For each $p\in P$, the notations f_p and ∂_xf are as in (2.1) and (2.2) respectively. Hence

$$\operatorname{gph}(\partial_xf)=\{(x,p,x^*)\in X\times P\times X^*:\ x^*\in\partial f_p(x)\}.$$

DEFINITION 3.1. *Given $q \in (1,+\infty)$ and $(\bar{x},\bar{p},\bar{x}^*) \in \text{gph}(\partial_x f)$, $\bar{x}$ is called*

(i) *a fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$ with modulus $\kappa > 0$ if there exist $\delta_1,\delta_2,r \in (0,+\infty)$ and a single-valued mapping $\vartheta : B_{X^*}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2) \to B_X(\bar{x},r)$ with $\vartheta(\bar{x}^*,\bar{p}) = \bar{x}$ such that*
(3.2)
$$\kappa\|x-\vartheta(u^*,p)\|^q \le f(x,p) - f(\vartheta(u^*,p),p) - \langle u^*, x-\vartheta(u^*,p)\rangle \quad \forall x \in B_X(\bar{x},r)$$
whenever $(u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2)$;

(ii) *a c-fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$ with modulus $\kappa > 0$ if, additionally, $\lim_{p\to\bar{p}} \vartheta(\bar{x}^*,p) = \vartheta(\bar{x}^*,\bar{p})$;*

(iii) *a fully (resp., c-fully) stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$ if there exists $\kappa > 0$ such that $\bar{x}$ is a fully (resp., c-fully) stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$ with modulus κ.*

It is clear that if $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$ then it is a fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$. The following proposition shows that the converse holds under the continuity assumption on f.

PROPOSITION 3.1. *Let $q \in (1,\ +\infty)$, $f : \mathbb{R}^n \times P \to \mathbb{R}$ be a continuous function, and let $(\bar{x},\bar{p},\bar{x}^*) \in \text{gph}(\partial_x f)$. Then $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$ if and only if it is a fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$.*

Proof. Since the necessity part is trivial, it suffices to prove the sufficiency part. Suppose that there exist $r,\delta_1,\delta_2,\kappa \in (0,+\infty)$ and a mapping $\vartheta : B_{\mathbb{R}^n}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2) \to B_{\mathbb{R}^n}(\bar{x},r)$ with $\vartheta(\bar{x}^*,\bar{p}) = \bar{x}$ such that (3.2) holds whenever $(u^*,p) \in B_{\mathbb{R}^n}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2)$. We only need to show that $\lim_{p\to\bar{p}} \vartheta(\bar{x}^*,p) = \bar{x}$. To do this, suppose to the contrary that there exist a sequence $\{p_k\}$ in $B_P(\bar{p},\delta_2)$ and $\varepsilon_0 > 0$ such that

$$\lim_{k\to\infty} p_k = \bar{p} \quad \text{and} \quad \varepsilon_0 < \|\vartheta(\bar{x}^*,p_k) - \bar{x}\| \quad \forall k \in \mathbb{N}. \tag{3.3}$$

Since $\vartheta(\bar{x}^*,p_k) \in B_{\mathbb{R}^n}(\bar{x},r)$, (3.2) implies that

$$\begin{aligned}\kappa\|\vartheta(\bar{x}^*,p_k) &- \vartheta(\bar{x}^*,\bar{p})\|^q \\ &\le f(\vartheta(\bar{x}^*,p_k),\bar{p}) - f(\vartheta(\bar{x}^*,\bar{p}),\bar{p}) - \langle \bar{x}^*, \vartheta(\bar{x}^*,p_k) - \vartheta(\bar{x}^*,\bar{p})\rangle \quad \forall k \in \mathbb{N}\end{aligned}$$

and

$$\kappa\|\bar{x} - \vartheta(\bar{x}^*,p_k)\|^q \le f(\bar{x},p_k) - f(\vartheta(\bar{x}^*,p_k),p_k) - \langle \bar{x}^*, \bar{x} - \vartheta(\bar{x}^*,p_k)\rangle \quad \forall k \in \mathbb{N}.$$

Noting that $\vartheta(\bar{x}^*,\bar{p}) = \bar{x}$, it follows from (3.3) that

$$2\kappa\varepsilon_0^q \le f(\bar{x},p_k) - f(\bar{x},\bar{p}) + f(\vartheta(\bar{x}^*,p_k),\bar{p}) - f(\vartheta(\bar{x}^*,p_k),p_k) \quad \forall k \in \mathbb{N}. \tag{3.4}$$

Since $\{\vartheta(\bar{x}^*,p_k)\}$ is a bounded sequence in $\mathbb{R}^n$, we can assume without loss of generality that $\vartheta(\bar{x}^*,p_k) \to x_0 \in \mathbb{R}^n$ (taking a subsequence if necessary), and therefore $(\vartheta(\bar{x}^*,p_k),p_k) \to (x_0,\bar{p})$. This and the continuity of f imply that

$$\lim_{k\to\infty} (f(\bar{x},p_k) - f(\bar{x},\bar{p}) + f(\vartheta(\bar{x}^*,p_k),\bar{p}) - f(\vartheta(\bar{x}^*,p_k),p_k)) = 0,$$

contradicting (3.4). The proof is complete. ∎

Clearly, (3.2) implies that $\vartheta(u^*, p)$ is a unique minimizer of the function $x \mapsto f(x,p) - \langle u^*, x\rangle$ over $B_X(\bar{x}, r)$ for all $(u^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2)$. Moreover, replacing x in (3.2) by $\vartheta(v^*, p)$ and also using the symmetry between v^* and u^*, (3.2) implies clearly that

$$\begin{aligned} 2\kappa\|\vartheta(v^*,p) - \vartheta(u^*,p)\|^q &\le -\langle u^*, \vartheta(v^*,p) - \vartheta(u^*,p)\rangle - \langle v^*, \vartheta(u^*,p) - \vartheta(v^*,p)\rangle \\ &= \langle v^* - u^*, \vartheta(v^*,p) - \vartheta(u^*,p)\rangle \\ &\le \|v^* - u^*\|\|\vartheta(v^*,p) - \vartheta(u^*,p)\|, \end{aligned}$$

and so

(3.5)

$$2\kappa\|\vartheta(v^*,p) - \vartheta(u^*,p)\|^{q-1} \le \|v^* - u^*\| \quad \forall v^*, u^* \in B_{X^*}(\bar{x}^*, \delta_1) \text{ and } p \in B_P(\bar{p}, \delta_2).$$

Thus, (3.2) implies that the single variable mapping $u^* \mapsto \vartheta(u^*, p)$ is continuous on $B_{X^*}(\bar{x}^*, \delta_1)$ for all $p \in B_P(\bar{p}, \delta_2)$. However, (3.2) does not imply the (joint) continuity of the double variable mapping $(u^*, p) \mapsto \vartheta(u^*, p)$. This makes the following proposition meaningful.

PROPOSITION 3.2. *Let $q \in (1, +\infty)$, $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$, $\delta_1, \delta_2, r, \kappa \in (0, +\infty)$, and a mapping $\vartheta : B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2) \to B_X(\bar{x}, r)$ with $\vartheta(\bar{x}^*, \bar{p}) = \bar{x}$ be such that* (3.2) *holds. Suppose that $(u_0^*, p_0) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2)$ satisfies*

$$\lim_{p \to p_0} \vartheta(u_0^*, p) = \vartheta(u_0^*, p_0). \tag{3.6}$$

Then ϑ is continuous at (u_0^*, p_0).

Proof. Since (3.2) implies (3.5), $\lim_{u^* \to u_0^*} \vartheta(u^*, p) = \vartheta(u_0^*, p)$ holds uniformly with respect to $p \in B_P(\bar{p}, \delta_2)$. Noting that

$$\|\vartheta(u^*,p) - \vartheta(u_0^*,p_0)\| \le \|\vartheta(u^*,p) - \vartheta(u_0^*,p)\| + \|\vartheta(u_0^*,p) - \vartheta(u_0^*,p_0)\|,$$

it follows from (3.6) that $\lim_{(u^*,p) \to (u_0^*,p_0)} \vartheta(u^*, p) = \vartheta(u_0^*, p_0)$. This shows that ϑ is continuous at (u_0^*, p_0). ∎

The following corollary is immediate from Proposition 3.2 (applied to $(u_0^*, p_0) = (\bar{x}^*, \bar{p})$).

COROLLARY 3.1. *Let $q \in (1, +\infty)$ and $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$ be such that $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*, \bar{p})$. Then there exist $\delta_1, \delta_2, r, \kappa \in (0, +\infty)$ and a mapping $\vartheta : B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2) \to B_X(\bar{x}, r)$ with $\vartheta(\bar{x}^*, \bar{p}) = \bar{x}$ such that both* (3.2) *and*

$$\lim_{(u^*,p) \to (\bar{x}^*,\bar{p})} \vartheta(u^*, p) = \vartheta(\bar{x}^*, \bar{p}) \tag{3.7}$$

hold.

Motivated by the uniform second-order growth condition (cf. [1, 3, 4, 7, 8, 9, 11]), we adopt the following notion.

DEFINITION 3.2. *Let $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$ and let $q \in (1, +\infty)$. We say that f satisfies a uniform q-order growth condition at $(\bar{x}, \bar{p}, \bar{x}^*)$ if there exist $\kappa, r, \delta_1, \delta_2 \in (0, +\infty)$ such that*

$$\kappa\|x - u\|^q \le f(x,p) - f(u,p) - \langle u^*, x - u\rangle \quad \forall x \in B_X(\bar{x}, r) \tag{3.8}$$

whenever $(u, p, u^*) \in \mathrm{gph}(\partial_x f) \cap (B_X(\bar{x}, r) \times B_P(\bar{p}, \delta_2) \times B_{X^*}(\bar{x}^*, \delta_1))$.

Comparing Definition 3.2 with Definition 3.1, it is worth noting the following two remarks.

(1) We require (3.2) in Definition 3.1 to hold for all $(u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2)$, while inequality (3.8) in Definition 3.2 is only required to hold for those $(u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2)$ with $u^* \in \partial f_p(B_X(\bar{x},r))$.

(2) Given $(u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2)$, inequality (3.2) in Definition 3.1 is only required to hold for a single $u = \vartheta(u^*,p)$ in $B_X(\bar{x},r) \cap (\partial f_p)^{-1}(u^*)$; in contrast, inequality (3.8) in Definition 3.2 is required to hold for all $u \in B_X(\bar{x},r) \cap (\partial f_p)^{-1}(u^*)$.

Moreover, given $r,\delta_1,\delta_2 \in (0,\ +\infty)$ and $(u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2)$, it is clear that

$$\operatorname*{arg\,min}_{u\in B_X(\bar{x},r)} \{f(u,p) - \langle u^*,u\rangle\} = (\partial f_p)^{-1}(u^*) \cap B_X(\bar{x},r)$$

if (3.8) holds. However, even in the special case in which the parameter space P is a singleton and (3.2) holds, it may happen that $\arg\min_{x\in B_X(\bar{x},r)}\{f(x,p) - \langle u^*,x\rangle\}$, equal to the singleton $\{\vartheta(u^*,p)\}$, is properly contained in $(\partial f_p)^{-1}(u^*) \cap B_X(\bar{x},r)$, as the following example shows.

Example 3.1. Let $q \in (1,+\infty)$ and $\{a_n\}$, $\{b_n\}$ be sequences in $(0,+\infty)$ such that

$$a_1 = 1, \quad a_{n+1} < b_n < a_n \quad \forall n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Define $f : \mathbb{R} \to \overline{\mathbb{R}}$ as follows: $f(x) = +\infty$ for all $x \in \mathbb{R} \setminus [-a_1,\ a_1]$ and

$$f(x) := \begin{cases} 2a_n & \text{if } x \in [-a_n,-b_n) \text{ and } n \in \mathbb{N}, \\ \frac{2a_n-2a_{n+1}}{a_{n+1}-b_n}(x+a_{n+1}) + 2a_{n+1} & \text{if } x \in [-b_n,-a_{n+1}) \text{ and } n \in \mathbb{N}, \\ 0 & \text{if } x = 0, \\ \frac{2a_n-2a_{n+1}}{b_n-a_{n+1}}(x-a_{n+1}) + 2a_{n+1} & \text{if } x \in (a_{n+1},b_n] \text{ and } n \in \mathbb{N}, \\ 2a_n & \text{if } x \in (b_n,a_n] \text{ and } n \in \mathbb{N}. \end{cases}$$

Then, since $q > 1$, it is easy to verify that $f(x) \geq 2|x| > |x| + |x|^q$ for all $x \in B_{\mathbb{R}}(0,1) \setminus \{0\}$, and so

$$f(x) - \langle u^*,x\rangle > |x| - \langle u^*,x\rangle + |x|^q \geq |x|^q \quad \forall (x,u^*) \in (B_{\mathbb{R}}(0,1) \setminus \{0\}) \times B_{\mathbb{R}}(0,1).$$

This implies that

$$\operatorname*{arg\,min}_{x\in B_{\mathbb{R}}(0,1)} \{f(x) - \langle u^*,x\rangle\} = \{0\} \quad \forall u^* \in B_{\mathbb{R}}(0,1)$$

and $\bar{x} = 0$ is a (fully) stable q-order minimizer of f. On the other hand, it is easy to verify that $0 \in \partial f(0)$ and

$$\partial f(x) := \begin{cases} 0 & \text{if } x \in (-a_n,-b_n) \text{ and } n \in \mathbb{N}, \\ \left[\frac{2a_n-2a_{n+1}}{a_{n+1}-b_n}, 0\right] & \text{if } x = -a_{n+1} \text{ and } n \in \mathbb{N}, \\ \left[0, \frac{2a_n-2a_{n+1}}{b_n-a_{n+1}}\right] & \text{if } x = a_{n+1} \text{ and } n \in \mathbb{N}, \\ 0 & \text{if } x \in (b_n,a_n) \text{ and } n \in \mathbb{N}. \end{cases}$$

Consequently,

$$\bigcup_{n=1}^{\infty}[-a_n, -b_n) \cup (b_n, a_n] \subset (\partial f)^{-1}(0), \quad \{0\} \cup \{a_{n+1}, n \in \mathbb{N}\} \subset (\partial f)^{-1}(u^*) \quad \forall u^* \in (0,1)$$

and

$$\{0\} \cup \{-a_{n+1}, n \in \mathbb{N}\} \subset (\partial f)^{-1}(u^*) \quad \forall u^* \in (-1,0).$$

Therefore, $B_{\mathbb{R}}(0,\varepsilon) \cap (\partial f)^{-1}(u^*)$ is not a singleton for all $\varepsilon > 0$ and $u^* \in B_{\mathbb{R}}(0,1)$.

Based on the above remarks and example, we introduce the following notion.

DEFINITION 3.3. *We say that the function $f : X \times P \to \overline{\mathbb{R}}$ satisfies the S-condition at $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$ if for all sufficiently small $r > 0$ there exist $\delta_1, \delta_2 \in (0, +\infty)$ and a mapping $\vartheta : B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2) \to X$ such that*

$$(\partial f_p)^{-1}(u^*) \cap B_X(\bar{x}, r) = \{\vartheta(u^*, p)\} \quad \forall (u^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2). \tag{3.9}$$

As a byproduct, this S-condition implies clearly that

$$\lim_{(u^*,p)\to(\bar{x}^*,\bar{p})} \vartheta(u^*, p) = \bar{x}.$$

Moreover, (3.9) also implies that $\mathrm{gph}(\partial_x f) \cap (B_X(\bar{x}, r) \times B_P(\bar{p}, \delta_2) \times B_{X^*}(\bar{x}^*, \delta_1))$ is equal to the set $\{(u, p, \vartheta(u^*, p)) : (u^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2)\}$. Therefore, if f satisfies the S-condition at $(\bar{x}, \bar{p}, \bar{x}^*)$ and $\bar{x}$ is a fully stable q-order minimizer of f for $(\bar{x}^*, \bar{p})$, then f satisfies a uniform q-order growth condition at $(\bar{x}, \bar{p}, \bar{x}^*)$. In what follows, we consider the relationships among the fully stable q-order minimizer, uniform q-order growth condition, and S-condition.

PROPOSITION 3.3. *Let $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$ and $(q, \kappa) \in (1, +\infty) \times (0, +\infty)$ be such that $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*, \bar{p})$ with modulus κ. Suppose that f is continuously q-regular in x at $\bar{x}$ for $(\bar{x}^*, \bar{p})$ with the associated constant ρ such that $0 \le \rho < \kappa$. Then f satisfies the S-condition.*

Proof. By Corollary 3.1 and the assumption that $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*, \bar{p})$ with modulus κ, there exist $\delta_1, \delta_2, r \in (0, +\infty)$ and a single-valued mapping $\vartheta : B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2) \to B_X(\bar{x}, r)$ with $\vartheta(\bar{x}^*, \bar{p}) = \bar{x}$ such that (3.2) and (3.7) hold. Thus, we have

$$\vartheta(u^*, p) \in (\partial f_p)^{-1}(u^*) \cap B_X(\bar{x}, r) \quad \forall (u^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2). \tag{3.10}$$

Since f is continuously q-regular in x at $\bar{x}$ for $(\bar{x}^*, \bar{p})$ with constant ρ, we can assume without loss of generality that

$$\langle u^*, x - u\rangle \le f(x, p) - f(u, p) + \rho\|x - u\|^q \quad \forall x \in B_X(\bar{x}, r) \tag{3.11}$$

whenever $(u, p) \in B_X(\bar{x}, r) \times B_P(\bar{p}, \delta_2)$ and $u^* \in \partial_x f(u, p) \cap B_{X^*}(\bar{x}^*, \delta_1)$ (taking smaller r, δ_1, δ_2 if necessary). By (3.10), to prove that the S-condition holds, it suffices to show that

$$(\partial f_p)^{-1}(u^*) \cap B_X(\bar{x}, r) \subset \{\vartheta(u^*, p)\} \quad \forall (u^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2). \tag{3.12}$$

Let $(u^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2)$ and $x_{u^*,p} \in (\partial f_p)^{-1}(u^*) \cap B_X(\bar{x}, r)$. Then, $u^* \in \partial_x f(x_{u^*,p}, p)$, and so it follows from (3.11) and (3.2) that

$$\langle u^*, \vartheta(u^*, p) - x_{u^*,p}\rangle \le f(\vartheta(u^*, p), p) - f(x_{u^*,p}, p) + \rho\|\vartheta(u^*, p) - x_{u^*,p}\|^q$$

and

$$\kappa\|x_{u^*,p} - \vartheta(u^*,p)\|^q \le f(x_{u^*,p},p) - f(\vartheta(u^*,p),p) - \langle u^*, x_{u^*,p} - \vartheta(u^*,p)\rangle.$$

This implies that $(\kappa - \rho)\|\vartheta(u^*,p) - x_{u^*,p}\|^q \le 0$, and so $x_{u^*,p} = \vartheta(u^*,p)$ due to the assumption that $\rho < \kappa$. Thus (3.12) is shown and the proof is complete. ◻

Proposition 3.3 requires a quite restrictive assumption: the modulus constant ρ of the continuous q-regularity of f in x at $\bar{x}$ for $(\bar{x}^*,\bar{p})$ is smaller than the modulus κ in (3.2). In the case in which X is a Hilbert space, such a restrictive assumption can be dropped.

PROPOSITION 3.4. *Let $q \in (1,+\infty)$ and $\bar{x}^* \in \partial_x f(\bar{x},\bar{p})$. Suppose that $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$ and that f is continuously prox-regular in x at $\bar{x}$ for $(\bar{x}^*,\bar{p})$. Further suppose that X is a Hilbert space. Then f satisfies the S-condition at $(\bar{x},\bar{p},\bar{x}^*)$. Consequently f satisfies a uniform q-order growth condition at $(\bar{x},\bar{p},\bar{x}^*)$.*

Proof. By Corollary 3.1, there exist $\delta_1,\delta_2,r,\kappa \in (0,+\infty)$ and a mapping $\vartheta : B_{X^*}(\bar{x}^*,\delta_1)\times B_P(\bar{p},\delta_2) \to B_X(\bar{x},r)$ with $\vartheta(\bar{x}^*,\bar{p}) = \bar{x}$ such that (3.2) and (3.7) hold. Take an r_0 in $(0,\ r)$. Thus, by (3.7), we can assume without loss of generality that

$$\|\vartheta(u^*,p) - \bar{x}\| < r_0 \quad \forall (u^*,p) \in B_X(\bar{x}^*,\delta_1)\times B_P(\bar{p},\delta_2). \tag{3.13}$$

Moreover, (3.2) implies that

$$f_p(\vartheta(u^*,p)) - \langle u^*, \vartheta(u^*,p)\rangle = \min_{x\in B_X[\bar{x},r_0]}(f_p(x) - \langle u^*,x\rangle)$$

for all $(u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1)\times B_P(\bar{p},\delta_2)$, where $f_p := f(\cdot,p)$. It follows that

$$(f_p + \delta_{B_X[\bar{x},r_0]})^*(u^*) = \langle u^*, \vartheta(u^*,p)\rangle - f_p(\vartheta(u^*,p)) \tag{3.14}$$

for all $(u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1)\times B_P(\bar{p},\delta_2)$, where $(f_p + \delta_{B_X[\bar{x},r_0]})^*$ denotes the conjugate function of $f_p + \delta_{B_X[\bar{x},r_0]}$. Let g_p denote the convex envelope of $f_p + \delta_{B_X[\bar{x},r_0]}$, that is,

$$\mathrm{epi}(g_p) = \overline{\mathrm{co}}(\mathrm{epi}(f_p + \delta_{B_X[\bar{x},r_0]})).$$

Then, $(f_p + \delta_{B_X[\bar{x},r_0]})^* = g_p^*$ and $g_p \le f_p + \delta_{B_X[\bar{x},r_0]}$. It follows from (3.14) and (3.13) that

$$\begin{aligned}\langle u^*, \vartheta(u^*,p)\rangle - f_p(\vartheta(u^*,p)) = g_p^*(u^*) &\ge \langle u^*, \vartheta(u^*,p)\rangle - g_p(\vartheta(u^*,p))\\ &\ge \langle u^*, \vartheta(u^*,p)\rangle - (f_p + \delta_{B_X[\bar{x},r_0]})(\vartheta(u^*,p))\\ &= \langle u^*, \vartheta(u^*,p)\rangle - f_p(\vartheta(u^*,p))\end{aligned}$$

and so

$$g_p^*(u^*) = \langle u^*, \vartheta(u^*,p)\rangle - g_p(\vartheta(u^*,p)) \quad \text{and} \quad f_p(\vartheta(u^*,p)) = g_p(\vartheta(u^*,p)) \tag{3.15}$$

for all $(u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1)\times B_P(\bar{p},\delta_2)$; thus

$$\vartheta(u^*,p) \in \partial g_p^*(u^*) \quad \forall (u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1)\times B_P(\bar{p},\delta_2).$$

By (3.5) and [14, Proposition 2.8], the convex function g_p^* is smooth on $B_{X^*}(\bar{x},\delta_1)$ and

$$\nabla g_p^*(u^*) = \{\vartheta(u^*,p)\} \quad \forall (u^*,p) \in B_{X^*}(\bar{x}^*,\delta_1)\times B_P(\bar{p},\delta_2),$$

which means

$$\text{gph}(\partial g_p^*) \cap (B_{X^*}(\bar{x}^*, \delta_1) \times X) = \{(u^*, \vartheta(u^*, p)) :\ u^* \in B_{X^*}(\bar{x}^*, \delta_1)\} \quad \forall p \in B_P(\bar{p}, \delta_2).$$

Since $\text{gph}(\partial g_p^*) = \{(x^*, x) :\ (x, x^*) \in \text{gph}(\partial g_p)\}$, it follows from (3.13) that

$$\text{gph}(\partial g_p) \cap (B_X(\bar{x}, r_0) \times B_{X^*}(\bar{x}^*, \delta_1)) = \{(\vartheta(u^*, p), u^*) :\ u^* \in B_{X^*}(\bar{x}, \delta_1)\} \tag{3.16}$$

for all $p \in B_P(\bar{p}, \delta_2)$. Noting that $g_p(x) \le f_p(x) + \delta_{B_X[\bar{x}, r_0]}(x) = f_p(x)$ for all $x \in B_X(\bar{x}, r_0)$, it follows from the convexity of g_p and (3.15) that

$$\langle u^*, x - \vartheta(u^*, p)\rangle \le g_p(x) - g_p(\vartheta(u^*, p)) \le f_p(x) - f_p(\vartheta(u^*, p))$$

for all $(x, u^*, p) \in B_X(\bar{x}, r_0) \times B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2)$. This and (3.13) imply that

$$u^* \in \hat{\partial} f_p(\vartheta(u^*, p)) \subset \partial f_p(\vartheta(u^*, p)) \quad \forall (u^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2).$$

Thus, by (3.16), one has

$$\text{gph}(\partial g_p) \cap (B_X(\bar{x}, r_0) \times B_{X^*}(\bar{x}^*, \delta_1)) \subset \text{gph}(\partial f_p) \quad \forall p \in B_P(\bar{p}, \delta_2). \tag{3.17}$$

On the other hand, by the continuous prox-regularity assumption on f, we can assume without loss of generality that there exists $\sigma \in (0, +\infty)$ such that

$$\langle u^*, x - u\rangle \le f(x, p) - f(u, p) + \frac{\sigma}{2}\|x - u\|^2 \quad \forall x \in B_X(\bar{x}, r_0) \tag{3.18}$$

whenever $(u, p) \in B_X(\bar{x}, r_0) \times B_P(\bar{p}, \delta_2)$ and $u^* \in \partial_x f(u, p) \cap B_{X^*}(\bar{x}^*, \delta_1)$ (taking smaller r_0, δ_1, and δ_2 if necessary). It follows that

$$0 \le \langle x^* - u^*, x - u\rangle + \sigma\|x - u\|^2 = \langle x^* + \sigma x - (u^* + \sigma u), x - u\rangle \tag{3.19}$$

for all $(x, x^*), (u, u^*) \in \text{gph}(\partial f_p) \cap (B_X(\bar{x}, r_0) \times B_X(\bar{x}^*, \delta_1))$ and $p \in B_P(\bar{p}, \delta_2)$. Take $r_0' \in (0,\ r_0)$ and $\delta_1' \in (0,\ \delta_1)$ such that $\delta_1' + \sigma r_0' < \delta_1$. Then, $x^* \in B_{X^*}(\bar{x}^*, \delta_1)$ whenever $x \in B_X(\bar{x}, r_0')$ and $x^* + \sigma x \in B_{X^*}(\bar{x}^* + \sigma\bar{x}, \delta_1')$. Thus, by (3.19) and (3.17),

$$\text{gph}(\partial f_p + \sigma I) \cap (B_X(\bar{x}, r_0') \times B_{X^*}(\bar{x}^* + \sigma\bar{x}, \delta_1')) \text{ is monotone for all } p \in B_P(\bar{p}, \delta_2) \tag{3.20}$$

and

$$\text{gph}(\partial g_p + \sigma I) \cap (B_X(\bar{x}, r_0') \times B_{X^*}(\bar{x}^* + \sigma\bar{x}, \delta_1')) \subset \text{gph}(\partial f_p + \sigma I) \quad \forall p \in B_P(\bar{p}, \delta_2). \tag{3.21}$$

Let $r' := \frac{1}{4}\min\{r_0',\ \frac{\delta_1'}{\sigma}\}$ and $\eta \in (0,\ r')$. Then, by (3.7), there exist

$$\bar{\delta}_1 \in (0,\ \min\{2\sigma(r' - \eta), \delta_1\}) \quad \text{and} \quad \bar{\delta}_2 \in (0,\ \delta_2)$$

such that

$$\|\vartheta(u^*, p) - \bar{x}\| < \eta \quad \forall (u^*, p) \in B_{X^*}(\bar{x}^*, \bar{\delta}_1) \times B_P(\bar{p}, \bar{\delta}_2). \tag{3.22}$$

It suffices to show that

$$(\partial f_p)^{-1}(u^*) \cap B_X(\bar{x}, \eta) = \{\vartheta(u^*, p)\} \quad \forall (u^*, p) \in B_{X^*}(\bar{x}^*, \bar{\delta}_1) \times B_P(\bar{p}, \bar{\delta}_2). \tag{3.23}$$

From (3.22), the definition of r' and the choice of η, it is easy to verify that

$$B_X(\bar{x},\eta) \subset B_X(\vartheta(\bar{x}^*,p),2r') \subset B_X(\bar{x},r_0') \quad \forall p \in B_P(\bar{p},\bar{\delta}_2) \tag{3.24}$$

and

$$B_{X^*}(\bar{x}^*+\sigma\vartheta(\bar{x}^*,p),2\sigma r') \subset B_{X^*}(\bar{x}^*+\sigma\bar{x},\delta_1') \quad \forall p \in B_P(\bar{p},\bar{\delta}_2). \tag{3.25}$$

For each p, let

$$W_p := B_X(\vartheta(\bar{x}^*,p),2r') \times B_{X^*}(\bar{x}^*+\sigma\vartheta(\bar{x}^*,p),2\sigma r').$$

Then, since $\bar{x}^* \in \partial g_p(\vartheta(\bar{x}^*,p))$ and ∂g_p is maximally monotone for all $p \in B_P(\bar{p},\delta_2)$, Lemma 2.2 implies that $\mathrm{gph}(\partial g_p+\sigma I)\cap W_p$ is a maximally monotone subset of W_p when $p \in B_P(\bar{p},\delta_2)$. Thus, by (3.20), (3.21), (3.24), and (3.25), one has

$$\mathrm{gph}(\partial g_p+\sigma I)\cap W_p = \mathrm{gph}(\partial f_p+\sigma I)\cap W_p \quad \forall p \in B_P(\bar{p},\bar{\delta}_2).$$

It follows from (3.22) and (3.24) that

$$\begin{aligned}\mathrm{gph}(\partial g_p)\cap(B_X(\bar{x},\eta)\times B_{X^*}(\bar{x}^*,2\sigma(r'-\eta)))\\ = \mathrm{gph}(\partial f_p)\cap(B_X(\bar{x},\eta)\times B_{X^*}(\bar{x}^*,2\sigma(r'-\eta)))\end{aligned}$$

for all $p \in B_P(\bar{x},\bar{\delta}_2)$. Since $\bar{\delta}_1 < \min\{2\sigma(r'-\eta),\delta_1\}$, this and (3.16) imply that (3.23) holds. The proof is complete. □

From the proof of Proposition 3.4, we have the following result.

PROPOSITION 3.4′. *Let X be a Hilbert space and let $\bar{x}^* \in \partial_x f(\bar{x},\bar{p})$ be such that f is continuously prox-regular in x at $\bar{x}$ for $(\bar{x}^*,\bar{p})$. Suppose that there exist $\delta_1,\delta_2,r,\tau \in (0,+\infty)$ and a single-valued mapping $\vartheta : B_{X^*}(\bar{x}^*,\delta_1)\times B_P(\bar{p},\delta_2) \to B_X(\bar{x},r)$ such that $\lim_{(u^*,p)\to(\bar{x}^*,\bar{p})}\vartheta(u^*,p)=\vartheta(\bar{x}^*,\bar{p})=\bar{x}$, the mapping $u^*\mapsto\vartheta(u^*,p)$ is continuous on $B_{X^*}(\bar{x},\delta_1)$, and*

$$f(\vartheta(u^*,p),p)-\langle u^*,\vartheta(u^*,p)\rangle = \min_{x\in B_X(\bar{x},r)}(f(x,p)-\langle u^*,x\rangle)$$

for all $p \in B_P(\bar{p},\delta_2)$. Then f satisfies the S-condition at $(\bar{x},\bar{p},\bar{x}^*)$.

The $(\mathcal{BCQ})$ plays an important role as a basic constraint qualification in the work by Levy, Poliquin, and Rockafellar [5] and Mordukhovich and Nghia [8, 10, 11]. The notion given below is a weaker one and plays a role somewhat similar to that played by $(\mathcal{BCQ})$ in [5] and [8, 10, 11].

DEFINITION 3.4. *Given $s \in (0,+\infty)$, the objective function $f(x,p)$ is said to satisfy the s-order basic constraint qualification (in brief s-$(\mathcal{BCQ})$) at $(\bar{x},\bar{p})$ if there exist $L \in (0,\ +\infty)$, and neighborhoods U of $\bar{p}$ and G of $(\bar{x},f(\bar{x},\bar{p}))$ such that*

$$s\text{-}(\mathcal{BCQ}) \qquad \mathrm{epi}(f_{p_1})\cap G \subset \mathrm{epi}(f_{p_2})+Ld(p_1,p_2)^s(B_X\times[-1,\ 1]) \quad \forall p_1,p_2\in U;$$

$f(x,p)$ is said to satisfy the weak basic constraint qualification (weak $(\mathcal{BCQ})$) at $(\bar{x},\bar{p})$ if there exists $s \in (0,+\infty)$ such that $f(x,p)$ satisfies s-$(\mathcal{BCQ})$ at $(\bar{x},\bar{p})$.

The following example shows that $(\mathcal{BCQ})$ is strictly stronger than weak $(\mathcal{BCQ})$.

Example 3.2. Let X be a Banach space, $P = \mathbb{R}$, $s \in (0,\ 1)$, and define $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ to be such that $f(x,p) := \varphi(x) + |p|^s$ for all $(x,p) \in X \times P$, where $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function. Clearly,

$$\mathrm{epi}(f_p) = (0, |p|^s) + \mathrm{epi}(\varphi) \quad \forall p \in P. \tag{3.26}$$

Noting that

$$(a+b)^s \leq a^s + b^s \quad \forall a, b \in [0,\ +\infty)$$

(thanks to $s \in (0,\ 1)$), one has

$$\|(0, |p_1|^s) - (0, |p_2|^s)\| = ||p_1|^s - |p_2|^s| \leq |p_1 - p_2|^s \quad \forall p_1, p_2 \in P.$$

It follows from (3.26) that s-$(\mathcal{BCQ})$ holds for any $(\bar{x}, \bar{p}) \in \mathrm{dom}(f)$ with $L = 1$, $U = P$, and $G = X \times \mathbb{R}$. This shows that $f(x,p)$ satisfies the s-order basic constraint qualification at each $(\bar{x}, \bar{p})$ in $\mathrm{dom}(f)$. Next suppose that there exist $x_0 \in \mathrm{dom}(\varphi)$ and $r > 0$ such that $\varphi(x_0) = \min_{x \in B(x_0, r)} \varphi(x)$. Then, by (3.26), one has

$$(x_0, \varphi(x_0)) \in \mathrm{epi}(f_0) \setminus \mathrm{epi}(f_{\frac{1}{n}}) \quad \text{and} \quad d((x_0, \varphi(x_0)), \mathrm{epi}(f_{\frac{1}{n}})) = \frac{1}{n^s}$$

for all $n \in \mathbb{N}$ with $n > \frac{1}{r^{\frac{1}{s}}}$, and so

$$\lim_{n\to\infty} \frac{d((x_0, \varphi(x_0)), \mathrm{epi}(f_{\frac{1}{n}}))}{d(0, \frac{1}{n})} = \lim_{n\to\infty} n^{1-s} = +\infty.$$

It follows that for any $L \in (0,\ +\infty)$ there exists $n_L \in \mathbb{N}$ such that

$$(x_0, \varphi(x_0)) \notin \mathrm{epi}(f_{\frac{1}{n}}) + Ld\left(0, \frac{1}{n}\right)(B_X \times [-1,\ 1]) \quad \forall n \in \mathbb{N} \text{ with } n \geq n_L.$$

Hence

$$\mathrm{epi}(f_0) \cap G \not\subset \mathrm{epi}(f_{\frac{1}{n}}) + Ld\left(0, \frac{1}{n}\right)(B_X \times [-1,\ 1])$$

for any neighborhood G of $(x_0, f(x_0, 0))$ and $n \in \mathbb{N}$ with $n \geq n_L$. This shows that $f(x,p)$ does not satisfy the basic constraint qualification $(\mathcal{BCQ})$ at $(x_0, 0)$.

The following lemma is similar to [8, Proposition 3.1] and immediate from Definition 3.4.

LEMMA 3.1. *Let $s \in (0, +\infty)$ and $(\bar{x}, \bar{p}) \in \mathrm{dom}(f)$ be such that f satisfies s-$(\mathcal{BCQ})$ at $(\bar{x}, \bar{p})$. Then there exist $r, \delta_2, \varepsilon, \ell \in (0, +\infty)$ such that*

$$\left.\begin{array}{l} x_1 \in B_X(\bar{x}, r),\ p_1, p_2 \in B_P(\bar{p}, \delta_2), \\ f(x_1, p_1) \leq f(\bar{x}, \bar{p}) + \varepsilon \end{array}\right\} \Longrightarrow \exists x_2 \text{ with } \begin{cases} \|x_1 - x_2\| \leq \ell d(p_1, p_2)^s, \\ f(x_2, p_2) \leq f(x_1, p_1) + \ell d(p_1, p_2)^s. \end{cases} \tag{3.27}$$

The following is another lemma about the weak-$(\mathcal{BCQ})$ which plays an important role in the proofs of some results.

LEMMA 3.2. *Let* $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$ *be such that* f *satisfies the weak-*$(\mathcal{BCQ})$ *at* $(\bar{x}, \bar{p})$ *and suppose that there exist* $q \in (1, +\infty)$ *and* $\delta_0 > 0$ *such that*

$$\kappa\|x - \bar{x}\|^q \leq f(x, \bar{p}) - f(\bar{x}, \bar{p}) - \langle \bar{x}^*, x - \bar{x}\rangle \quad \forall x \in B_X[\bar{x}, \delta_0]. \tag{3.28}$$

Then, for all sufficiently small $\eta > 0$ *there exists* $\delta_\eta > 0$ *with the following property: for any* $(u^*, p) \in B_{X^*}(\bar{x}^*, \delta_\eta) \times B_P(\bar{p}, \delta_\eta)$ *there exist sequences* $\{v_n\} \subset X$ *and* $\{v_n^*\} \subset X^*$ *such that*

$$f(v_n, p) - \langle u^*, v_n\rangle \leq \inf_{x \in B_X[\bar{x}, \eta]} \{f(x, p) - \langle u^*, x\rangle\} + \frac{1}{n^2} \quad \forall n \in \mathbb{N}, \tag{3.29}$$

$$v_n^* \in B_{X^*}\left[u^*, \frac{1}{n}\right] \quad \text{and} \quad v_n \in (\partial f_p)^{-1}(v_n^*) \cap B_X\left(\bar{x}, \frac{2\eta}{3}\right) \quad \forall n \in \mathbb{N}. \tag{3.30}$$

Proof. By the weak-$(\mathcal{BCQ})$ assumption and Lemma 3.1, there exist $s, r, \delta_2, \varepsilon, \ell \in (0, +\infty)$ such that (3.27) holds. Since f is lower semicontinuous at $(\bar{x}, \bar{p}) \in \mathrm{dom}(f)$, f is bounded below on a neighborhood of $(\bar{x}, \bar{p})$. Hence there exist sufficiently small $\eta, \delta_\eta \in (0, +\infty)$ such that f is bounded below on $B_X[\bar{x}, \eta] \times B_P(\bar{p}, \delta_\eta)$,

$$\eta \leq \frac{\min\{r, \delta_2, \delta_0\}}{2}, \quad \delta_\eta \leq \min\left\{\frac{r}{2}, \delta_2\right\}, \quad \ell\delta_\eta^s \leq \frac{\eta}{3}, \quad 2\left(\|\bar{x}^*\| + \delta_\eta + \frac{7}{6}\right)\eta \leq \varepsilon, \tag{3.31}$$

and

$$2\ell\delta_\eta^s(\|\bar{x}^*\| + \delta_\eta + 1) + \delta_\eta\delta_0 < \kappa\left(\frac{\eta}{3}\right)^q. \tag{3.32}$$

Take an arbitrary (u^*, p) in $B_{X^*}(\bar{x}^*, \delta_\eta) \times B_P(\bar{p}, \delta_\eta)$. Then there exists a sequence $\{u_n\} \subset B_X[\bar{x}, \eta]$ such that

$$f(u_n, p) - \langle u^*, u_n\rangle \leq \inf_{x \in B_X[\bar{x}, \eta]} \{f(x, p) - \langle u^*, x\rangle\} + \frac{1}{n^2} \quad \forall n \in \mathbb{N}.$$

This and Ekeland's variational principle imply that for each $n \in \mathbb{N}$ there exists $v_n \in B_X[\bar{x}, \eta]$ such that (3.29) holds,

$$\|v_n - u_n\| \leq \frac{1}{n}, \tag{3.33}$$

and

$$f(v_n, p) - \langle u^*, v_n\rangle \leq f(x, p) - \langle u^*, x\rangle + \frac{1}{n}\|x - v_n\| \quad \forall x \in B_X[\bar{x}, \eta]. \tag{3.34}$$

By choosing $(x_1, p_1) = (\bar{x}, \bar{p})$ and $p_2 = p$ in (3.27), we can find $x_2 = w$ such that

$$\|w - \bar{x}\| \leq \ell d(p, \bar{p})^s \leq \ell\delta_\eta^s < \eta \tag{3.35}$$

and $f(\bar{x}, \bar{p}) \geq f(w, p) - \ell d(p, \bar{p})^s \geq f(w, p) - \ell\delta_\eta^s$. This and (3.34) imply that

$$\begin{aligned} f(\bar{x}, \bar{p}) + \ell\delta_\eta^s &\geqslant f(w, p) \\ &\geq f(v_n, p) + \langle u^*, w - v_n\rangle - \frac{1}{n}\|w - v_n\| \\ &\geq f(v_n, p) - \|u^*\| \cdot \|w - v_n\| - \|w - v_n\| \\ &\geqslant f(v_n, p) - (\|\bar{x}^*\| + \delta_\eta + 1)\|w - v_n\|. \end{aligned} \tag{3.36}$$

Since $\|w - v_n\| \le \|w - \bar{x}\| + \|\bar{x} - v_n\| < 2\eta$, it follows that

$$f(v_n, p) \le f(\bar{x}, \bar{p}) + \ell\delta_\eta^s + 2(\|\bar{x}^*\| + \delta_\eta + 1)\eta,$$

and so $f(v_n, p) \le f(\bar{x}, \bar{p}) + \varepsilon$ (thanks to (3.31)). Thus, setting $(x_1, p_1) = (v_n, p)$ and $p_2 = \bar{p}$ in (3.27), we can find $z_n \in X$ such that

$$\|z_n - v_n\| \le \ell d(p, \bar{p})^s \le \ell\delta_\eta^s \tag{3.37}$$

and

$$f(v_n, p) \ge f(z_n, \bar{p}) - \ell d(p, \bar{p})^s \ge f(z_n, \bar{p}) - \ell\delta_\eta^s. \tag{3.38}$$

It follows from (3.31) that

$$\|z_n - \bar{x}\| \le \|z_n - v_n\| + \|v_n - \bar{x}\| \le \ell\delta_\eta^s + \eta \le \frac{4\eta}{3} < \delta_0. \tag{3.39}$$

Moreover, by (3.36), (3.38), and (3.28), one has

$$\begin{aligned} f(\bar{x}, \bar{p}) + \ell\delta_\eta^s &\geqslant f(v_n, p) + \langle u^*, w - v_n\rangle - n^{-1}\|w - v_n\| \\ &\geqslant f(z_n, \bar{p}) - \ell\delta_\eta^s + \langle u^*, w - v_n\rangle - n^{-1}\|w - v_n\| \\ &\geqslant f(\bar{x}, \bar{p}) + \langle \bar{x}^*, z_n - \bar{x}\rangle + \kappa\|z_n - \bar{x}\|^q - \ell\delta_\eta^s + \langle u^*, w - v_n\rangle - 2n^{-1}\eta. \end{aligned}$$

This, together with (3.37), (3.35), and (3.39), implies that

$$\begin{aligned} 2\ell\delta_\eta^s + 2n^{-1}\eta &\geqslant \kappa\|z_n - \bar{x}\|^q + \langle \bar{x}^* - u^*, z_n - \bar{x}\rangle + \langle u^*, z_n - \bar{x} + w - v_n\rangle \\ &\geqslant \kappa\|z_n - \bar{x}\|^q - \|\bar{x}^* - u^*\| \cdot \|z_n - \bar{x}\| - \|u^*\|(\|z_n - v_n\| + \|w - \bar{x}\|) \\ &\geqslant \kappa\|z_n - \bar{x}\|^q - \delta_\eta\delta_0 - 2\ell\delta_\eta^s(\|\bar{x}^*\| + \delta_\eta), \end{aligned}$$

and so $\kappa\|z_n - \bar{x}\|^q \le 2\ell\delta_\eta^s(\|\bar{x}^*\| + \delta_\eta + 1) + \delta_\eta\delta_0 + 2n^{-1}\eta$. It follows from (3.32) that

$$\limsup_{n\to+\infty} \kappa\|z_n - \bar{x}\|^q \le 2\ell\delta_\eta^s(\|\bar{x}^*\| + \delta_\eta + 1) + \delta_\eta\delta_0 < \kappa\left(\frac{\eta}{3}\right)^q.$$

Hence, $\|z_n - \bar{x}\| < \frac{\eta}{3}$ for all sufficiently large n. Since $\|v_n - \bar{x}\| \le \|v_n - z_n\| + \|z_n - \bar{x}\|$, it follows from (3.37) and (3.31) that

$$v_n \in B_X\left(\bar{x}, \frac{2\eta}{3}\right) \text{ for all sufficiently large } n. \tag{3.40}$$

This and (3.34) imply that $0 \in \partial_x f(v_n, p) - u^* + n^{-1}B_{X^*}$, namely there exists $v_n^* \in B_{X^*}[u^*, \frac{1}{n}]$ such that $v_n \in (\partial f_p)^{-1}(v_n^*)$. Therefore, by (3.40), one can see that (3.30) holds. The proof is complete. ∎

THEOREM 3.1. *Let $f : X \times P \to \overline{\mathbb{R}}$ satisfy the weak-($\mathcal{BCQ}$) at $(\bar{x}, \bar{p}) \in \mathrm{dom}(f)$. Let $q \in (1, +\infty)$ and $\bar{x}^* \in \partial_x f(\bar{x}, \bar{p})$ be such that f satisfies the uniform q-order growth condition at $(\bar{x}, \bar{p}, \bar{x}^*)$. Then $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*, \bar{p})$ and f satisfies the S-condition at $(\bar{x}, \bar{p}, \bar{x}^*)$.*

Proof. Since f satisfies the uniform q-order growth condition at $(\bar{x}, \bar{p}, \bar{x}^*)$, there exist $\kappa, r, \delta_1, \delta_2 \in (0, +\infty)$ such that (3.8) holds for all

$$(u, p, u^*) \in \mathrm{gph}(\partial_x f) \cap (B_X(\bar{x}, r) \times B_P(\bar{p}, \delta_2) \times B_{X^*}(\bar{x}^*, \delta_1)).$$

Hence

$$\begin{cases}\kappa\|u_2-u_1\|^q \le f(u_2,p)-f(u_1,p)-\langle u_1^*, u_2-u_1\rangle,\\ \kappa\|u_2-u_1\|^q \le f(u_1,p)-f(u_2,p)-\langle u_2^*, u_1-u_2\rangle,\end{cases}$$

and so

$$2\kappa\|u_1-u_2\|^q \le \langle u_1^*-u_2^*, u_1-u_2\rangle \le \|u_1^*-u_2^*\|\cdot\|u_1-u_2\|$$

for all $(u_1,p,u_1^*),(u_2,p,u_2^*)\in \mathrm{gph}(\partial_x f)\cap(B_X(\bar{x},r)\times B_P(\bar{p},\delta_2)\times B_{X^*}(\bar{x}^*,\delta_1))$. This implies that

$$(\partial f_p)^{-1}(u^*)\cap B_X(\bar{x},r)=\{\vartheta(u^*,p)\} \tag{3.41}$$

and

$$(2\kappa)^{\frac{1}{q-1}}\|\vartheta(u_1^*,p)-\vartheta(u_2^*,p)\| \le \|u_1^*-u_2^*\|^{\frac{1}{q-1}} \tag{3.42}$$

for all $p\in B_P(\bar{p},\delta_2)$ and $u^*,u_1^*,u_2^*\in B_{X^*}(\bar{x}^*,\delta_1)\cap\partial f_p(B_X(\bar{x},r))$. Thus, by Definitions 3.1 and 3.3 and (3.8), it suffices to show that there exists $\delta'\in(0,\ \min\{\delta_1,\delta_2\})$ such that

$$B_{X^*}(\bar{x}^*,\delta')\subset\partial f_p(B_X(\bar{x},r))\quad \forall p\in B_P(\bar{p},\delta'). \tag{3.43}$$

To do this, let η be an arbitrary number in $(0,\ r)$. Then, by (3.8), the weak-($\mathcal{BCQ}$) assumption, and Lemma 3.2, there exists $\delta_\eta\in(0,\ \min\{\delta_1,\delta_2\})$ with the following property: given any $(u^*,p)\in B_{X^*}(\bar{x}^*,\delta_\eta)\times B_P(\bar{p},\delta_\eta)$, there exist sequences $\{v_n\}\subset X$ and $\{v_n^*\}\subset X^*$ such that (3.29) and (3.30) hold. Hence, by (3.30), (3.41), and (3.42), we have that $v_n=\vartheta(v_n^*,p)\in B_X(\bar{x},\frac{2\eta}{3})$ for all $n\in\mathbb{N}$, $\lim_{n\to\infty}v_n^*=u^*$, $\{v_n\}$ is a Cauchy sequence, and so $\lim_{n\to\infty}v_n\to v\in B_X[\bar{x},\frac{2\eta}{3}]$. It follows from (3.29) that $f(v,p)-\langle u^*,v\rangle=\min_{x\in B_X[\bar{x},\eta]}(f(x,p)-\langle u^*,x\rangle)$ and hence $0\in\partial_x f(v,p)-u^*$. This implies that $u^*\in\partial f_p(v)\subset\partial f_p(B_X(\bar{x},r))$, and so (3.43) holds with $\delta'=\delta_\eta$. The proof is complete. ∎

The following lemma is useful for our analysis later, which is established in the proof of [17, Theorem 4.3].

LEMMA 3.3. *Let X be a Banach space and $\varphi: X\to\overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x}\in\mathrm{dom}(\varphi)$ and $r_1>0$ be such that $\varphi(\bar{x})=\min_{x\in B_X[\bar{x},r_1]}\varphi(x)$. Suppose that $\partial\varphi$ is strongly γ-order metrically regular at $(\bar{x},0)$ (with $\gamma\in(0,\ +\infty)$), in the sense that there exist $r_2,\kappa,\delta\in(0,\ +\infty)$ and a mapping $\vartheta: B_{X^*}(0,\delta)\to X$ with $\vartheta(0)=\bar{x}$ such that*

$$(\partial\varphi)^{-1}(u^*)\cap B_X(\bar{x},r_2)=\{\vartheta(u^*)\}\quad\text{and}\quad \kappa\|\vartheta(u^*)-\vartheta(v^*)\|\le\|u^*-v^*\|^\gamma \tag{3.44}$$

for all $u^*,v^*\in B_{X^*}(0,\delta)$. Then

$$\tau\|x-\vartheta(u^*)\|^{\frac{1+\gamma}{\gamma}}\le\varphi(x)-\varphi(\vartheta(u^*))-\langle u^*,x-\vartheta(u^*)\rangle\quad\forall(x,u^*)\in B_X(\bar{x},\eta')\times B_{X^*}(0,\delta'), \tag{3.45}$$

where

$$\tau:=\frac{\gamma\kappa^{\frac{1}{\gamma}}}{(1+\gamma)^{\frac{1+\gamma}{\gamma}}},\quad \eta':=\frac{1+\gamma}{4(1+2\gamma)}\min\{r_1,r_2,\delta\},$$

and

$$\delta':=\min\left\{\frac{\eta'}{2},\left(\frac{2\kappa\eta'}{1+2\gamma}\right)^{\frac{1}{\gamma}},\tau\left(\frac{\eta'}{2}\right)^{\frac{1}{\gamma}}\right\}.$$

With the help of Lemma 3.3, we can establish the following sufficient condition for f to have a c-fully stable Hölder minimizer.

THEOREM 3.2. *Let $(\bar{x},\bar{p},\bar{x}^*) \in \mathrm{gph}(\partial_x f)$ be such that $\bar{x}$ is a local minimizer of the function $f_{\bar{p}} - \bar{x}^*$ and f satisfies the weak-($\mathcal{BCQ}$) at $(\bar{x},\bar{p})$. Suppose that $\partial_x f$ is uniformly strongly $\frac{1}{q-1}$-order metrically regular at $(\bar{x},\bar{p},\bar{x}^*)$ (with $q > 1$), in the sense that there exist $r,\kappa,\delta_1,\delta_2 \in (0,+\infty)$ and a mapping $\vartheta : B_{X^*}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2) \to B_X(\bar{x},r)$ with $\vartheta(\bar{x}^*,\bar{p}) = \bar{x}$ such that*

(3.46)
$$(\partial f_p)^{-1}(u^*) \cap B_X(\bar{x},r) = \{\vartheta(u^*,p)\} \quad \text{and} \quad \kappa\|\vartheta(u^*,p) - \vartheta(v^*,p)\| \le \|u^* - v^*\|^{\frac{1}{q-1}}$$

for all $u^*, v^* \in B_{X^*}(\bar{x}^*,\delta_1)$ and $p \in B_P(\bar{p},\delta_2)$. Then f satisfies the uniform q-order growth condition at $(\bar{x},\bar{p},\bar{x}^*)$. Consequently, $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$.

Proof. Since $\bar{x}$ is a local minimizer of $f_{\bar{p}} - \bar{x}^*$, it follows from (3.46) and Lemma 3.3 that there exist $\bar{\tau} \in (0,\ +\infty)$ and $r' \in (0,\ r)$ such that

$$\bar{\tau}\|x - \bar{x}\|^q \le f(x,\bar{p}) - f(\bar{x},\bar{p}) - \langle \bar{x}^*, x - \bar{x}\rangle \quad \forall x \in B_X[\bar{x},r'].$$

Thus, by Lemma 3.2, there exists $\bar{r} \in (0,\ \frac{r'}{2})$ such that for any $\varepsilon \in (0,\ 2\bar{r}]$ there exists $\delta_\varepsilon \in (0,\ \delta_2)$ with the following property: for any $p \in B_P(\bar{p},\delta_\varepsilon)$ there exist sequences $\{v_n\} \subset X$ and $\{v_n^*\} \subset X^*$ such that

$$f(v_n,p) - \langle \bar{x}^*, v_n\rangle \le \inf_{x\in B_X[\bar{x},\varepsilon]}\{f(x,p) - \langle \bar{x}^*, x\rangle\} + \frac{1}{n^2} \quad \forall n \in \mathbb{N}, \tag{3.47}$$

$$v_n^* \in B_{X^*}\left[\bar{x}^*, \frac{1}{n}\right] \quad \text{and} \quad v_n \in (\partial f_p)^{-1}(v_n^*) \cap B_X\left(\bar{x}, \frac{2\varepsilon}{3}\right) \quad \forall n \in \mathbb{N}.$$

This and (3.46) imply that

$$v_n = \vartheta(v_n^*,p) \to \vartheta(\bar{x}^*,p) \in B_X\left[\bar{x}, \frac{2\varepsilon}{3}\right] \quad \forall p \in B_P(\bar{p},\delta_\varepsilon). \tag{3.48}$$

Hence $\lim_{p\to\bar{p}} \vartheta(\bar{x}^*,p) = \bar{x}$ (because ε is arbitrary in $(0,\ 2\bar{r}]$). Thus, by (3.46), we have that

$$\lim_{(u^*,p)\to(\bar{x}^*,\bar{p})} \vartheta(u^*,p) = \bar{x} \tag{3.49}$$

and that there exists $\tilde{\delta} \in (0,\ \min\{\delta_1,\delta_2\})$ such that

$$\|\vartheta(u^*,p) - \bar{x}\| < \frac{r}{2} \quad \text{and} \quad (\partial f_p)^{-1}(u^*) \cap B_X\left(\vartheta(\bar{x}^*,p), \frac{r}{2}\right) = \{\vartheta(u^*,p)\} \tag{3.50}$$

for all $(u^*,p) \in B_{X^*}(\bar{x}^*,\tilde{\delta}) \times B_P(\bar{p},\tilde{\delta})$. Since f is lower semicontinuous, (3.47) and (3.48) imply that

$$f(\vartheta(\bar{x}^*,p),p) - \langle \bar{x}^*, \vartheta(\bar{x}^*,p)\rangle = \inf_{x\in B_X[\bar{x},\bar{r}]}\{f(x,p) - \langle \bar{x}^*, x\rangle\} \quad \forall p \in B_P(\bar{p},\bar{\delta}),$$

where $\bar{\delta} := \delta_{\bar{r}}$. Thus, by Lemma 3.3 (with φ and $\bar{x}$ replaced by $f_p - \bar{x}^*$ and $\vartheta(\bar{x}^*,p)$, respectively), (3.46), and (3.50), there exist $\tau,\eta',\delta_1' \in (0,\ +\infty)$ (only dependent on $\kappa,\bar{r},\tilde{\delta}$ and $\bar{\delta}$) such that

$$\tau\|x - \vartheta(u^*,p)\|^q \le f(x,p) - f(\vartheta(u^*,p)) - \langle u^*, x - \vartheta(u^*,p)\rangle$$
$$\forall (x,u^*) \in B_X(\vartheta(\bar{x}^*,p),\eta') \times B_{X^*}(\bar{x}^*,\delta_1')$$

whenever $p \in B_P(\bar{p}, \min\{\tilde{\delta}, \bar{\delta}\})$. Thus, from (3.46) and (3.49), it is easy to verify that f satisfies the uniform q-order growth condition at $(\bar{x}, \bar{p}, \bar{x}^*)$. The proof is hence complete. ∎

4. Stable (q, s)-minimum with respect to both tilt and parameter perturbations. In this section, motivated by [5, 8, 10, 11], we consider the more general Hölder stability of a minimum with respect to both tilt and parameter perturbations. Given $r > 0$, let

$$M_r(u^*, p) := \operatorname*{arg\,min}_{x \in B_X(\bar{x}, r)} \{f(x, p) - \langle u^*, x\rangle\} \quad \forall (u^*, p) \in X^* \times P \tag{4.1}$$

and

$$m_r(u^*, p) := \inf_{x \in B_X(\bar{x}, r)} \{f(x, p) - \langle u^*, x\rangle\} \quad \forall (u^*, p) \in X^* \times P.$$

DEFINITION 4.1. *Given $q, s \in (0, +\infty)$ and $(\bar{x}, \bar{p}, \bar{x}^*) \in \operatorname{gph}(\partial_x f)$, we say that f gives a fully stable (q, s)-minimum at $\bar{x}$ for $(\bar{x}^*, \bar{p})$ if there exist $r, L \in (0, +\infty)$, a neighborhood W of $(\bar{x}^*, \bar{p})$ such that $(u^*, p) \mapsto M_r(u^*, p)$ is single valued on W with $M_r(\bar{x}^*, \bar{p}) = \bar{x}$ and*

$$\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\|^q \le L(\|u_1^* - u_2^*\|\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\| + d(p_1, p_2)^s) \tag{4.2}$$

for all $(u_1^*, p_1), (u_2^*, p_2) \in W$.

Inequality (4.2) is just (1.6) in section 1. Here we adopt the terminology "fully stable (q, s)-minimum" because it reduces to the tilt-stable q-minimum in [17] (and tilt-stable minimum in [15]) when P is a singleton (and $q = 2$).

Remark. Since $M_r(W) \subset B_X(\bar{x}, r)$, inequality (4.2) implies that

$$\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\|^q \le L'(\|u_1^* - u_2^*\| + d(p_1, p_2)^s) \quad \forall (u_1^*, p_1), (u_2^*, p_2) \in W, \tag{4.3}$$

where $L' := L \max\{2r, 1\}$. On the other hand, multiplying both sides of inequality (4.3) by $\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\|$, one has

$$\begin{aligned}
&\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\|^{q+1} \\
&\quad \le L'(\|u_1^* - u_2^*\|\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\| + 2rd(p_1, p_2)^s) \\
&\quad \le L''(\|u_1^* - u_2^*\|\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\| + d(p_1, p_2)^s)
\end{aligned}$$

for all $(u_1^*, p_1), (u_2^*, p_2) \in W$ with $L'' = L' \max\{2r, 1\}$. Next we prove that inequality (4.3) holds for some positive number L' if and only if there exists $K > 0$ such that

$$\|M_r(u_1^*, p_1) - M_r(u_2^*, p_2)\| \le K(\|u_1^* - u_2^*\|^{\frac{1}{q}} + d(p_1, p_2)^{\frac{s}{q}}) \quad \forall (u_1^*, p_1), (u_2^*, p_2) \in W. \tag{4.4}$$

Indeed, noting that

$$\begin{aligned}
(\|u_1^* - u_2^*\| + d(p_1, p_2)^s)^{\frac{1}{q}} &\le (2\max\{\|u_1^* - u_2^*\|, d(p_1, p_2)^s\})^{\frac{1}{q}} \\
&\le 2^{\frac{1}{q}}(\|u_1^* - u_2^*\|^{\frac{1}{q}} + d(p_1, p_2)^{\frac{s}{q}}),
\end{aligned}$$

(4.3) implies (4.4) with $K = (2L')^{\frac{1}{q}}$. On the other hand, since

$$(\|u_1^* - u_2^*\|^{\frac{1}{q}} + d(p_1, p_2)^{\frac{s}{q}})^q \le 2^q(\|u_1^* - u_2^*\| + d(p_1, p_2)^s),$$

(4.4) implies

$$\|M_r(u_1^*,p_1)-M_r(u_2^*,p_2)\|^q \le (2K)^q(\|u_1^*-u_2^*\|+d(p_1,p_2)^s) \quad \forall (u_1^*,p_1),(u_2^*,p_2)\in W,$$

and so (4.4) also implies (4.3) with $L'=(2K)^q$.

In the case in which $q=2$, we have the following exact relationship.

PROPOSITION 4.1. *Let $s\in(0,+\infty)$ and $(\bar{x},\bar{p},\bar{x}^*)\in \mathrm{gph}(\partial_x f)$. Then f gives a fully stable $(2,s)$-minimum at $\bar{x}$ for $(\bar{x}^*,\bar{p})$ if and only if there exist $r,l\in(0,\ +\infty)$, a neighborhood W of $(\bar{x}^*,\bar{p})$ such that $(u^*,p)\mapsto M_r(u^*,p)$ is single valued on W with $M_r(\bar{x}^*,\bar{p})=\bar{x}$ and*

$$\|M_r(u_1^*,p_1)-M_r(u_2^*,p_2)\| \le l(\|u_1^*-u_2^*\|+d(p_1,p_2)^{\frac{s}{2}}) \quad \forall (u_1^*,p_1),(u_2^*,p_2)\in W.$$

Proof. Since $q=2$, (4.2) can be rewritten as

$$\left(\|M_r(u_1^*,p_1)-M_r(u_2^*,p_2)\|-\frac{L}{2}\|u_1^*-u_2^*\|\right)^2 \le \frac{L^2}{4}\|u_1^*-u_2^*\|^2+Ld(p_1,p_2)^s,$$

and so (4.2) is equivalent to

$$\|M_r(u_1^*,p_1)-M_r(u_2^*,p_2)\| \le \frac{L}{2}\|u_1^*-u_2^*\|+\sqrt{\frac{L^2}{4}\|u_1^*-u_2^*\|^2+Ld(p_1,p_2)^s}.$$

Thus the proposition is shown because, elementarily,

$$\begin{aligned} l_1\|u_1^*-u_2^*\|+l_2 d(p_1,p_2)^{\frac{s}{2}} &\le \sqrt{\frac{L^2}{4}\|u_1^*-u_2^*\|^2+Ld(p_1,p_2)^s} \\ &\le L_1\|u_1^*-u_2^*\|+L_2 d(p_1,p_2)^{\frac{s}{2}} \end{aligned}$$

for all $(u_1^*,p_1),(u_2^*,p_2)\in X^*\times P$ and some $l_1,l_2,L_1,L_2\in(0,\ +\infty)$. ◻

In light of Proposition 4.1, we note that a fully stable $(2,2)$-minimum means a fully Lipschitz-stable minimum in the sense of Levy, Poliquin, and Rockafellar [5] while a fully stable $(2,1)$-minimum means a fully Hölder-stable minimum in the sense of Mordukhovich and Nghia [8]. The following example shows that Definition 4.1 genuinely extends the notions of both a fully Lipschitz-stable minimum in the sense of Levy, Poliquin, and Rockafellar [5] and a fully Hölder-stable minimum in the sense of Mordukhovich and Nghia [8].

Example 4.1. Let $X=P=\mathbb{R}$, $n\in\mathbb{N}\setminus\{1\}$, and $s\in(0,\ \frac{1}{2})$, and define $f: X\times P\to\mathbb{R}$ as follows:

$$f(x,p):=\frac{1}{2n}(x-|p|^s)^{2n}+a(p) \quad \forall (x,p)\in X\times P,$$

where $a(p)$ is a real-valued function. Clearly $x\mapsto f(x,p)$ is a smooth convex function for each $p\in P$. For any $u^*\in\mathbb{R}$, the convex function $x\mapsto f(x,p)-\langle u^*,x\rangle$ is smooth and

$$\arg\min_{x\in X}\{f(x,p)-\langle u^*,x\rangle\}=\{x\in\mathbb{R}:\nabla f(\cdot,p)(x)-u^*=0\}=\{u^{*\frac{1}{2n-1}}+|p|^s\} \quad \forall p\in P.$$

Setting $(\bar{x},\bar{x}^*,\bar{p})=(0,0,0)$ and letting $M(u^*,p):=u^{*\frac{1}{2n-1}}+|p|^s$, by the known inequality

$$(a+b)^\gamma \le a^\gamma+b^\gamma \quad \forall a,b\in(0,\ +\infty) \text{ and } \gamma\in(0,\ 1),$$

it is easy to verify that

$$\|M(u_1^*, p_1) - M(u_2^*, p_2)| \le |u_1^{*\frac{1}{2n-1}} - u_2^{*\frac{1}{2n-1}}| + ||p_1|^s - |p_2|^s| \\ \le |u_1^* - u_2^*|^{\frac{1}{2n-1}} + |p_1 - p_2|^s$$

for all $(u_1^*, p_1), (u_2^*, p_2) \in B(0,1) \times B(0,1)$. This shows that f gives a fully stable $(\frac{1}{2n-1}, s)$-minimum at $\bar{x}$ for $(\bar{x}^*, \bar{p})$. On the other hand,

$$\lim_{u^* \to 0} \frac{|M(u^*, 0) - M(0,0)|}{|u^* - 0|} = \lim_{u^* \to 0} |u^*|^{-\frac{2n-2}{2n-1}} = +\infty,$$

$$\lim_{p \to 0} \frac{|M(0,p) - M(0,0)|}{|p|^{\frac{1}{2}}} = \lim_{p \to 0} |p|^{s - \frac{1}{2}} = +\infty,$$

and

$$\lim_{p \to 0} \frac{|M(0,p) - M(0,0)|}{|p|} = \lim_{p \to 0} |p|^{s-1} = +\infty$$

(thanks to $s \in (0, \frac{1}{2})$). This shows that f gives neither a fully Lipschitz-stable minimum in the sense of Levy, Poliquin, and Rockafellar [5] nor a fully Hölder-stable minimum in the sense of Mordukhovich and Nghia [8].

Mordukhovich and Nghia (cf. [8, Proposition 4.2]) showed that the fully stable $(2,1)$-minimum and $(\mathcal{BCQ})$ imply the Lipschitz continuity of m_r. Similarly, the following proposition establishes the corresponding Hölder continuity of m_r for the more general fully stable (q, s')-minimum and s-$(\mathcal{BCQ})$.

PROPOSITION 4.2. *Let $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$ and $q, s', s \in (0, +\infty)$ be such that f gives a fully stable (q, s')-minimum at $\bar{x}$ for $(\bar{x}^*, \bar{p})$ and f satisfies s-$(\mathcal{BCQ})$ at $(\bar{x}, \bar{p})$. Then there exist $r, L \in (0, +\infty)$ and a neighborhood W of $(\bar{x}^*, \bar{p})$ such that*

$$|m_r(u_2^*, p_2) - m_r(u_1^*, p_1)| \le L(\|u_2^* - u_1^*\| + d(p_1, p_2)^s) \quad \forall (u_1^*, p_1), (u_2^*, p_2) \in W. \tag{4.5}$$

Proof. By the assumption, there exist $r, L, \delta \in (0, +\infty)$ such that $(u^*, p) \mapsto M_r(u^*, p)$ is single valued on $B_{X^*}(\bar{x}^*, \delta) \times B_P(\bar{p}, \delta)$, with $M_r(\bar{x}^*, \bar{p}) = \bar{x}$, and (4.2) holds (with $s = s'$) for all $(u^*, p) \in B_{X^*}(\bar{x}^*, \delta) \times B_P(\bar{p}, \delta)$. Hence

$$\begin{aligned} \|M_r(u^*, p) - \bar{x}\|^q &= \|M_r(u^*, p) - M_r(\bar{x}^*, \bar{p})\|^q \\ &\le L(\|u^* - \bar{x}^*\| \cdot \|M_r(u^*, p) - \bar{x}\| + d(p, \bar{p})^{s'}) \\ &\le rL\|u^* - \bar{x}^*\| + Ld(p, \bar{p})^{s'} \end{aligned} \tag{4.6}$$

for all $(u^*, p) \in B_{X^*}(\bar{x}^*, \delta) \times B_P(\bar{p}, \delta)$. It follows from s-$(\mathcal{BCQ})$ and Lemma 3.1 that we can assume that there exist $\ell, \delta_2, \varepsilon \in (0, +\infty)$ such that (3.27) holds (noting that r in Lemma 3.1 can be replaced by an arbitrarily small one). By (4.6), take $r_0 \in (0, r)$ and $\eta \in (0, \min\{\delta, \delta_2\})$ such that

$$M_r(B_{X^*}(\bar{x}^*, \eta) \times B_P(\bar{p}, \eta)) \subset B_X(\bar{x}, r_0), \tag{4.7}$$

$$\ell(2\eta)^s + r_0 < r, \quad \text{and} \quad \ell\eta^s + (\|\bar{x}^*\| + \eta)(r_0 + \ell\eta^s) < \varepsilon. \tag{4.8}$$

Take any $(u_1^*, p_1), (u_2^*, p_2) \in B_{X^*}(\bar{x}^*, \eta) \times B_P(\bar{p}, \eta)$ and let $u_1 := M_r(u_1^*, p_1)$ and $u_2 := M_r(u_2^*, p_2)$. Then $u_1, u_2 \in B_X(\bar{x}, r_0)$. By (3.27), there exists $w \in X$ such that

$$\|w - \bar{x}\| \le \ell d(p_1, \bar{p})^s \le \ell\eta^s < r \quad \text{and} \quad f(w, p_1) \le f(\bar{x}, \bar{p}) + \ell d(p_1, \bar{p})^s.$$

Hence

$$f(u_1,p_1) - \langle u_1^*, u_1\rangle \le f(w,p_1) - \langle u_1^*, w\rangle \le f(\bar{x},\bar{p}) + \ell d(p_1,\bar{p})^s - \langle u_1^*, w\rangle. \tag{4.9}$$

It follows from the second inequality of (4.8) that

$$\begin{aligned} f(u_1,p_1) &\le f(\bar{x},\bar{p}) + \ell d(p_1,\bar{p})^s + \|u_1^*\|\|u_1 - w\| \\ &\le f(\bar{x},\bar{p}) + \ell\eta^s + (\|\bar{x}^*\| + \eta)(r_0 + \ell\eta^s) \\ &< f(\bar{x},\bar{p}) + \varepsilon. \end{aligned}$$

By (3.27) again, we could find $\tilde{u}_1 \in X$ such that

$$\|\tilde{u}_1 - u_1\| \le \ell d(p_1,p_2)^s \le \ell(2\eta)^s \quad \text{and} \quad f(\tilde{u}_1,p_2) \le f(u_1,p_1) + \ell d(p_1,p_2)^s.$$

Thus, by the first inequality of (4.8), one has $\|\tilde{u}_1 - \bar{x}\| \le \|\tilde{u}_1 - u_1\| + \|u_1 - \bar{x}\| \le \ell(2\eta)^s + r_0 < r$. It follows from the definitions of u_1 and u_2 that

$$\begin{aligned} m_r(u_2^*,p_2) - m_r(u_1^*,p_1) &= f(u_2,p_2) - \langle u_2^*, u_2\rangle - (f(u_1,p_1) - \langle u_1^*, u_1\rangle) \\ &\le f(\tilde{u}_1,p_2) - \langle u_2^*, \tilde{u}_1\rangle - (f(u_1,p_1) - \langle u_1^*, u_1\rangle) \\ &\le \ell d(p_1,p_2)^s + \langle u_1^*, u_1\rangle - \langle u_2^*, \tilde{u}_1\rangle. \end{aligned} \tag{4.10}$$

Since

$$\begin{aligned} \langle u_1^*, u_1\rangle - \langle u_2^*, \tilde{u}_1\rangle &= \langle u_1^* - u_2^*, u_1\rangle - \langle u_2^*, \tilde{u}_1 - u_1\rangle \\ &\le (\|\bar{x}\| + r_0)\|u_2^* - u_1^*\| + (\|\bar{x}^*\| + \eta)\|u_1 - \tilde{u}_1\| \\ &\le (\|\bar{x}\| + r_0)\|u_2^* - u_1^*\| + \ell(\|\bar{x}^*\| + \eta)d(p_1,p_2)^s, \end{aligned}$$

we have

$$m_r(u_2^*,p_2) - m_r(u_1^*,p_1) \le (\|\bar{x}\| + r_0)\|u_2^* - u_1^*\| + \ell(1 + \|\bar{x}^*\| + \eta)d(p_1,p_2)^s.$$

Changing the role of (u_2^*,p_2) and (u_1^*,p_1) in the above inequality, one has

$$|m_r(u_2^*,p_2) - m_r(u_1^*,p_1)| \le (\|\bar{x}\| + r_0)\|u_2^* - u_1^*\| + \ell(1 + \|\bar{x}^*\| + \eta)d(p_1,p_2)^s,$$

verifying that (4.5) holds with $L := \max\{\|\bar{x}\| + r_0, \ell(1 + \|\bar{x}^*\| + \eta)\}$. ☐

THEOREM 4.1. *Let $q \in (1,+\infty)$, $s \in (0, +\infty)$ and $(\bar{x},\bar{p},\bar{x}^*) \in \mathrm{gph}(\partial_x f)$. Suppose that f satisfies s-($\mathcal{BCQ}$) at $(\bar{x},\bar{p})$ and that $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$. Then f gives a fully stable (q,s)-minimum at $\bar{x}$ for $(\bar{x}^*,\bar{p})$.*

Proof. By Corollary 3.1 and the assumption that $\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*,\bar{p})$, there exist $\delta_1,\delta_2,r,\kappa \in (0,+\infty)$ and a mapping $\vartheta : B_{X^*}(\bar{x}^*,\delta_1) \times B_P(\bar{p},\delta_2) \to B_X(\bar{x},r)$ with $\vartheta(\bar{x}^*,\bar{p}) = \bar{x}$ such that both (3.2) and (3.7) hold. Since f satisfies s-($\mathcal{BCQ}$) at $(\bar{x},\bar{p})$, by Lemma 3.1, we assume without loss of generality that there exist $\varepsilon,\ell \in (0, +\infty)$ such that (3.27) holds (taking smaller r and δ_2 if necessary). Hence there exist $\eta \in (0, \frac{r}{2})$ and $\delta \in (0, \min\{\delta_1,\delta_2\})$ such that

$$\|\vartheta(u^*,p) - \bar{x}\| < \eta \quad \forall (u^*,p) \in B_{X^*}(\bar{x}^*,\delta) \times B_P(\bar{p},\delta), \tag{4.11}$$

$$0 < \ell(2\delta)^s < \eta < \frac{r}{2}, \quad \text{and} \quad \ell\delta^s + 2(\|\bar{x}^*\| + \delta + 1)\eta < \varepsilon. \tag{4.12}$$

Take any $(u_i^*, p_i) \in B_{X^*}(\bar{x}^*, \delta) \times B_P(\bar{p}, \delta)$ and let $u_i := \vartheta(u_i^*, p_i)$ $(i = 1, 2)$. Then, by (3.27) (with (x_1, p_1) and p_2 being replaced respectively by $(\bar{x}, \bar{p})$ and p_i), there exists $\tilde{x}_i \in X$ such that

$$\|\tilde{x}_i - \bar{x}\| \leq \ell d(p_i, \bar{p})^s < \ell \delta^s < \eta \quad \text{and} \quad f(\tilde{x}_i, p_i) \leq f(\bar{x}, \bar{p}) + \ell d(p_i, \bar{p})^s \quad (i = 1, 2).$$

This and (3.2) imply that

$$\begin{aligned} f(\bar{x}, \bar{p}) + \ell \delta^s &\geqslant f(\tilde{x}_i, p_i) \\ &\geq f(u_i, p_i) + \langle u_i^*, \tilde{x}_i - u_i \rangle + \kappa \|\tilde{x}_i - u_i\|^q \\ &\geqslant f(u_i, p_i) - \|u_i^*\| \cdot \|\tilde{x}_i - u_i\| \\ &\geq f(u_i, p_i) - 2(\|\bar{x}^*\| + \delta)\eta \end{aligned}$$

(the last inequality holds because of (4.11)). Hence

$$f(u_i, p_i) \leq f(\bar{x}, \bar{p}) + \ell \delta^s + 2(\|\bar{x}^*\| + \delta)\eta$$

and it follows from (4.12) that $f(u_i, p_i) \leq f(\bar{x}, \bar{p}) + \varepsilon$. By (3.27) again, there exist v_1, v_2 such that

$$\text{(4.13)} \quad \begin{cases} \|u_2 - v_1\| \leq \ell d(p_1, p_2)^s \leq \ell(2\delta)^s, \ \|u_1 - v_2\| \leq \ell d(p_1, p_2)^s \leq \ell(2\delta)^s, \\ f(v_1, p_1) \leq f(u_2, p_2) + \ell d(p_1, p_2)^s, \ f(v_2, p_2) \leq f(u_1, p_1) + \ell d(p_1, p_2)^s. \end{cases}$$

Hence, by the first inequality of (4.12), one has

$$\|v_1 - \bar{x}\| \leq \|v_1 - u_2\| + \|u_2 - \bar{x}\| \leq \ell(2\delta)^s + \eta < r,$$

which yields $v_1 \in B_X(\bar{x}, r)$; similarly, one has $v_2 \in B_X(\bar{x}, r)$. This, together with (3.2), ensures that

$$\begin{cases} \kappa \|v_1 - u_1\|^q \leq f(v_1, p_1) - f(u_1, p_1) - \langle u_1^*, v_1 - u_1 \rangle, \\ \kappa \|v_2 - u_2\|^q \leq f(v_2, p_2) - f(u_2, p_2) - \langle u_2^*, v_2 - u_2 \rangle, \end{cases}$$

and so

$$\begin{aligned} &\kappa \|v_1 - u_1\|^q + \kappa \|v_2 - u_2\|^q \\ &\quad \leq f(v_1, p_1) - f(u_2, p_2) + f(v_2, p_2) - f(u_1, p_1) - \langle u_1^*, v_1 - u_1 \rangle - \langle u_2^*, v_2 - u_2 \rangle. \end{aligned}$$

It follows from (4.13) that

$$\text{(4.14)} \quad \begin{aligned} &\kappa \|v_1 - u_1\|^q + \kappa \|v_2 - u_2\|^q \\ &\quad \leq 2\ell d(p_1, p_2)^s - \langle u_1^*, v_1 - u_1 \rangle - \langle u_2^*, v_2 - u_2 \rangle \\ &\quad = 2\ell d(p_1, p_2)^s - \langle u_1^*, v_1 - u_2 \rangle - \langle u_2^*, v_2 - u_1 \rangle - \langle u_1^* - u_2^*, u_2 - u_1 \rangle \\ &\quad \leq 2\ell d(p_1, p_2)^s + (\|u_1^*\| + \|u_2^*\|)\ell d(p_1, p_2)^s + \|u_2^* - u_1^*\| \|u_2 - u_1\| \\ &\quad \leq 2(\|\bar{x}^*\| + \delta + 1)\ell d(p_1, p_2)^s + \|u_2^* - u_1^*\| \|u_2 - u_1\|. \end{aligned}$$

Since the function $t \mapsto t^q$ is convex on $[0, +\infty)$,

$$\begin{aligned} \|v_1 - u_1\|^q - \|u_2 - u_1\|^q &\geq q\|u_2 - u_1\|^{q-1}(\|v_1 - u_1\| - \|u_2 - u_1\|) \\ &\geq -q\|u_2 - u_1\|^{q-1}\|v_1 - u_2\|. \end{aligned}$$

Hence, by (4.11) and (4.13), one has $\|v_1 - u_1\|^q \geq \|u_1 - u_2\|^q - q\ell(2\eta)^{q-1}d(p_1, p_2)^s$. Similarly, one also has $\|v_2 - u_2\|^q \geq \|u_1 - u_2\|^q - q\ell(2\eta)^{q-1}d(p_1, p_2)^s$. It follows from (4.14) that

$$\|u_1 - u_2\|^q \leq \frac{1}{2\kappa}\|u_1^* - u_2^*\| \cdot \|u_1 - u_2\| + \left[q\ell(2\eta)^{q-1} + \frac{\ell}{\kappa}(\|\bar{x}^*\| + \delta + 1)\right] d(p_1, p_2)^s.$$

This shows that f gives a fully stable (q, s)-minimum at $\bar{x}$ for $(\bar{x}^*, \bar{p})$. ∎

In the framework of Hilbert spaces, the following theorem provides some characterizations for f to give a fully stable (q, s)-minimum at $\bar{x}$ for $(\bar{x}^*, \bar{p})$.

THEOREM 4.2. *Let X be a Hilbert space and $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $q \in (1, +\infty)$, $s \in (0, +\infty)$, and $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$ be such that f is continuously prox-regular and satisfies s-$(\mathcal{BCQ})$ at $(\bar{x}, \bar{p}, \bar{x}^*)$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is a c-fully stable q-order minimizer of f for $(\bar{x}^*, \bar{p})$.*

(ii) *f satisfies a uniform q-order growth condition at $(\bar{x}, \bar{p}, \bar{x}^*)$.*

(iii) *f gives a fully stable (q, s)-minimum at $\bar{x}$ for $(\bar{x}^*, \bar{p})$.*

(iv) *$\bar{x}$ is a local minimizer of $f_{\bar{p}} - \bar{x}^*$ and there exist $r, \delta_1, \delta_2, L \in (0, +\infty)$ and a mapping $\vartheta : B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2) \to B_{X^*}(\bar{x}, r)$ with $\vartheta(\bar{x}^*, \bar{p}) = \bar{x}$ such that*

(4.15)
$$(\partial f_p)^{-1}(u^*) \cap B_{X^*}(\bar{x}, r) = \{\vartheta(u^*, p)\} \quad \forall (u^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2)$$

and

(4.16)
$$\|\vartheta(u_1^*, p_1) - \vartheta(u_2^*, p_2)\|^q \leq L(\|u_1^* - u_2^*\| \cdot \|\vartheta(u_1^*, p_1) - \vartheta(u_2^*, p_2)\| + d(p_1, p_2)^s)$$

for all $(u_1^*, p_1), (u_2^*, p_2) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2)$.

(v) *$\bar{x}$ is a local minimizer of $f_{\bar{p}} - \bar{x}^*$ and there exist $r, \delta_1, \delta_2, L \in (0, +\infty)$ and a mapping $\vartheta : B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2) \to B_{X^*}(\bar{x}, r)$ with $\vartheta(\bar{x}^*, \bar{p}) = \bar{x}$ such that (4.15) holds and*

$$\|\vartheta(u_1^*, p) - \vartheta(u_2^*, p)\| \leq L\|u_1^* - u_2^*\|^{\frac{1}{q-1}} \quad \forall (u_1^*, p), (u_2^*, p) \in B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2). \tag{4.17}$$

Proof. The implication (iv)⇒(v) is trivial, (ii)⇔(i)⇒(iii) are immediate from Proposition 3.4 and Theorems 3.1 and 4.1, while (v)⇒(ii) is just Theorem 3.2. Thus, we only need to show that (iii)⇒(iv). Now suppose that (iii) holds. Then there exist $r, L \in (0, +\infty)$, a neighborhood W of $(\bar{x}^*, \bar{p})$ such that $(u^*, p) \mapsto M_r(u^*, p)$ is single valued on W with $M_r(\bar{x}^*, \bar{p}) = \bar{x}$ and (4.2) holds for all $(u_1^*, p_1), (u_2^*, p_2) \in W$. It follows from Proposition 3.4′ that f satisfies the S-condition at $(\bar{x}, \bar{x}^*, \bar{p})$. Hence, by (4.1), there exists a neighborhood W_0 of $(\bar{x}^*, \bar{p})$ such that $W_0 \subset W$ and

$$\mathrm{gph}(\partial_x f) \cap (B_X(\bar{x}, r) \times W_0) = \{(u, p, M_r(u^*, p)) : (u^*, p) \in W_0\}.$$

Take $\delta_1, \delta_2 \in (0, +\infty)$ such that $B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2) \subset W_0$, and define $\vartheta(u^*, p) := M_r(u^*, p)$ for $B_{X^*}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2)$. Then, by (4.2), (4.15) and (4.16) hold. This shows (iii)⇒(iv). The proof is completed. ∎

In Theorem 4.1 and Theorem 4.2(i), the required $\lim_{p \to \bar{p}} \vartheta(\bar{x}^*, p) = \vartheta(\bar{x}^*, \bar{p})$ as a part of the c-fully stable q-order minimizer (see Definition 3.1) is somewhat restrictive. However, under the assumption that X is finite dimensional and f is continuous, Proposition 3.1 shows that "a c-fully stable q-order minimizer" can be relaxed to "a fully stable q-order minimizer" in Theorems 4.1 and 4.2.

Without any need to involve "order s of parameter p," we have the following corollary, which follows immediately from Theorem 4.2, Proposition 3.1, and the definition of weak-($\mathcal{BCQ}$).

COROLLARY 4.1. *Let $f : \mathbb{R}^n \times P \to \mathbb{R}$ be a continuous function, and let $q \in (1, +\infty)$ and $(\bar{x}, \bar{p}, \bar{x}^*) \in \mathrm{gph}(\partial_x f)$ be such that f is continuously prox-regular and satisfies weak-($\mathcal{BCQ}$) at $(\bar{x}, \bar{p}, \bar{x}^*)$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is a fully stable q-order minimizer of f for $(\bar{x}^*, \bar{p})$.*

(ii) *f satisfies a uniform q-order growth condition at $(\bar{x}, \bar{p}, \bar{x}^*)$.*

(iii) *$\bar{x}$ is a local minimizer of $f_{\bar{p}} - \bar{x}^*$ and there exist $r, \delta_1, \delta_2, L \in (0, +\infty)$ and a mapping $\vartheta : B_{\mathbb{R}^n}(\bar{x}^*, \delta_1) \times B_P(\bar{p}, \delta_2) \to B_{\mathbb{R}^n}(\bar{x}, r)$ with $\vartheta(\bar{x}^*, \bar{p}) = \bar{x}$ such that* (4.15) *and* (4.17) *hold.*

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