MATH2050B Mathematical Analysis I

Homework 4 suggested Solution*

Question 3*. Show that $\lim_n \frac{n^{100}}{(1+\epsilon)^n} = 0$, $\forall \epsilon > 0$.

Solution:

For $n \geq 101$, we note that

$$\begin{aligned} \frac{n^{100}}{(1+\epsilon)^n} &= \frac{n^{100}}{1 + C_1^n \epsilon + C_2^n \epsilon^2 + \cdots + C_k^n \epsilon^k + \cdots + \epsilon^n} \\ &\leq \frac{n^{100}}{C_{101}^n \epsilon^{101}} \\ &= \frac{101! \cdot n^{100}}{n(n-1)(n-2)\cdots(n-100)\epsilon^{101}} \\ &= \frac{101!}{n(1-1/n)(1-2/n)\cdots(1-100/n)\epsilon^{101}}. \end{aligned} \tag{1}$$

Notice that $\lim_{n\to\infty} \frac{101!}{n(1-1/n)(1-2/n)\cdots(1-100/n)\epsilon^{101}} = 0$ and $0 \leq \frac{n^{100}}{(1+\epsilon)^n}$, by computation rules, we thus get $\lim_{n\to\infty} \frac{n^{100}}{(1+\epsilon)^n} = 0$.

Question 5*. Let a>0. We know that $\sqrt{a}$ exists in $(0, \infty)$ (can you do this ?). Below is a "pratical way" to show not only its existence but also a correponding approximation/numerical procedure. Pick $x_1 > 0$ such that $x_1^2 > a$, and define

$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(i) Show that if $x := \lim_n x_n$ exists in $(0, \infty)$, then $x = \frac{a}{x}$ (and so x is the (positive) Sq. root of a). *By (ii) & (iii) below, the limit x does exist !*

(ii) Show that $x_n^2 \geqslant a, \quad \forall n \in \mathbb{N}$.

(Hint: $x_{n+1}^2 = \frac{1}{4}\left(x_n^2 + 2a + \left(\frac{a}{x_n}\right)^2\right) \geqslant a$ because $\left(x_n - \frac{a}{x_n}\right)^2 \geqslant 0$)

(iii) $(x_n) \downarrow$ (Hint: by (ii), $\frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \leqslant x_n$)

Solution:

*please kindly send an email to cyma@math.cuhk.edu.hk if you have any question.

(i) We first show that if $x := \lim_n x_n$ exists in $(0, \infty)$, then $\lim_n \frac{a}{x_n} = \frac{a}{x}$. To see this, we notice that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\frac{x}{2} \leq x_n, \quad \text{and} \quad |x_n - x| < \frac{x^2\epsilon}{2a}.$$

It follows that for any $n \geq N$,

$$\left|\frac{a}{x_n} - \frac{a}{x}\right| = \left|\frac{a(x_n - x)}{x_n x}\right| \leq \left|\frac{a(x_n - x)}{x^2/2}\right| = \frac{2a}{x^2}|x_n - x| < \frac{2a}{x^2}\frac{x^2\epsilon}{2a} = \epsilon.$$

Thus we have $\lim_n \frac{a}{x_n} = \frac{a}{x}$. Since $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, let n tend to infinity, we obtain

$$\lim_n x_{n+1} = \frac{1}{2}\left(\lim_n x_n + \lim_n \frac{a}{x_n}\right),$$

that is, $x = \dfrac{x}{2} + \dfrac{a}{2x}$, hence that $x = \dfrac{a}{x}$.

(ii) Let $P(n)$ denote the proposition that $x_n^2 \geq a$.

Notice that when $n = 1$, we have $x_1^2 \geq a$, thus $P(1)$ is true.

Suppose $P(n)$ is true, i.e. $x_n^2 \geq a$. It follows that

$$x_{n+1}^2 = \frac{1}{4}\left(x_n^2 + 2a + \left(\frac{a}{x_n}\right)^2\right) \geqslant \frac{a}{2} + \frac{1}{4}\left(x_n^2 + \frac{a^2}{x_n^2}\right) \geq \frac{a}{2} + \frac{a}{2} = a.$$

Hence $P(n+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. Therefore, we have $x_n^2 \geqslant a$ for all $n \in \mathbb{N}$.

(iii) It is easily seen that $x_n > 0$ for all $n \in \mathbb{N}$, since $x_1 > 0$. It follows from (ii) that $\frac{a}{x_n} \leq \frac{x_n^2}{x_n}$. Hence we get

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \leq \frac{x_n + x_n}{2} = x_n,$$

which implies that the sequence $\{x_n\}$ is a decreasing sequence.

Question 6*. Suppose $\lim_n x_n = 6$. Show in the $\epsilon - N$ terminology (and definitions only) that

$$\lim_n \frac{x_n^3 + 4}{x_n - 5} = 220.$$

(properties of $\mathbb{R}$ are allowed).

Solution:

Fix $\epsilon > 0$, since $\lim_n x_n = 6$, there exists $N_1(\epsilon) \in \mathbb{N}$ such that for any $n \geq N_1(\epsilon)$,

$$|x_n - 6| < \epsilon/200.$$

Besides there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have

$$|x_n - 6| < \frac{1}{2} \quad \text{i.e.} \quad \frac{11}{2} < x_n < \frac{13}{2},$$

which implies that $x_n - 5 > \frac{1}{2}$ and $|x_n| < 7$.

Denote $N = \max\{N_1(\epsilon), N_2\}$, then for any $n \geq N$,

$$\begin{aligned}
\left|\frac{x_n^3 + 4}{x_n - 5} - 220\right| &= \left|\frac{x_n^3 + 4}{x_n - 5} - \frac{6^3 + 4}{6 - 5}\right| \\
&= \left|\frac{1 \cdot (x_n^3 + 4) - (x_n - 5)(6^3 + 4)}{(x_n - 5)(6 - 5)}\right| \\
&\leq 2\left|x_n^3 + 4 - (x_n - 5)(6^3 + 4)\right| \\
&\leq 2\left|x_n^3 + 4 - (x_n - 5)6^3 - (x_n - 5)4\right| \\
&\leq 2\left|x_n^3 - 6^3 - (x_n - 6)6^3 - (x_n - 6)4\right| \\
&= 2\left|(x_n - 6)(x_n^2 + 6x_n + 6^2 - 6^3 - 4)\right| \\
&\leq 2\left|(x_n - 6)\right| \cdot |7^2 + 6 \times 7 + 6^2 - 6^3 - 4| \\
&= 2 \cdot 93\left|(x_n - 6)\right| \\
&\leq 2 \cdot 93\frac{\epsilon}{200} \leq \epsilon.
\end{aligned}$$

This shows that $\lim_n \frac{x_n^3+4}{x_n-5} = 220$.