MATH2050B Mathematical Analysis I

Homework 2 suggested Solution*

Question 3. Without assuming III, show that $-\sup X = \inf(-X)$, provided that either $\sup X$ exists in $\mathbb{R}$ or $\inf(-X)$ exists in $\mathbb{R}$.

Solution:

Suppose $\sup X$ exists, denoted by a. Then $x \leq a$ for any $x \in X$. Thus we have $-x \geq -a$ for all $x \in X$, which implies $-a$ is a lower bound of $-X$.

Moreover, if u is an upper bound of X, then $a \leq u$. Notice that u is an upper bound of X if and only if $-u$ is lower bound of $-X$. Therefore, $-a \geq -u$ for any lower bound $-u$ of $-X$. It follows by the defintion of $\inf(-X)$ that $\inf(-X) = -a = -\sup(X)$.

The case $\inf(-X)$ exists can be handled in much the same way, we omit the proof here.

Question 4. Let $\emptyset \neq A, B \subseteq \mathbb{R}$ and

$$A + B := \{a + b : a \in A, b \in B\}.$$

Show that

$$\sup(A + B) = \sup(A) + \sup(B),$$

provided that $\sup(A)$ and $\sup(B)$ exist in $\mathbb{R}$.

Solution:

Assume that $\alpha = \sup A$ and $\beta = \sup B$. Then $a + b \leq \alpha + \beta$, for any $a \in A, b \in B$. It follows that

$$\sup(A + B) \leq \alpha + \beta. \tag{1}$$

On the other hand, we note that $\sup(A + B) \geq a + b$, for any $a \in A, b \in B$. Hence, for any $b \in B$,

$$\sup(A + B) - b \geq a, \qquad \text{for all } a \in A.$$

It yields that $\sup(A + B) - b \geq \alpha$, since $(\sup(A + B) - b)$ is an upper bound of A. It follows that $\sup(A + B) - \alpha \geq b$, for all $b \in B$. Thus we have

$$\sup(A + B) - \alpha \geq \beta. \tag{2}$$

*please kindly send an email to cyma@math.cuhk.edu.hk if you have any question.

Combining inequalities (1) and (2), we get $\sup(A+B) = \sup(A) + \sup(B)$.

Question 5. Let $f, g : D \to \mathbb{R}$ be functions such that

$$\sup\{f(x)+g(x) : x \in D\} \quad \sup\{f(x) : x \in D\}, \quad \text{and } \sup\{g(x) : x \in D\}$$

exist in $\mathbb{R}$. Show that

$$\sup\{f(x)+g(x) : x \in D\} \leq \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\},$$

and provide a counter-example, by showing that " $\leq$ " cannot be replaced by " $=$ ".

Solution:

Assume that $a = \sup\{f(x) : x \in D\}$ and $b = \sup\{g(x) : x \in D\}$. It follows by the definition that

$$f(x) + g(x) \leq a + g(x) \leq a + b, \text{ for all } x \in D.$$

Therefore, we have $\sup\{f(x)+g(x) : x \in D\} \leq a + b$.

Next we give an example that $\sup\{f(x)+g(x) : x \in D\} < \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}$.

Let $f(x) = \sin x$ and $g(x) = -\sin x$, where $x \in [-\pi, \pi]$. Then

$$\sup\{f(x) : x \in [-\pi,\pi]\} = \sup\{g(x) : x \in [-\pi,\pi]\} = 1;$$

$$\sup\{f(x)+g(x) : x \in [-\pi,\pi]\} = 0.$$

Therefore, " $\leq$ " cannot be replaced by " $=$ ".

Question 8. "Solve" the inequality system:

$$(\sharp) \qquad 4 < |x+2| + |x-1| \leq 5,$$

that is, let X consist of all x satisfying the above inequalities, express X.

Hint: Try to remove the absolute value signs seperately in each of the following cases:

(i) both $(x+2)$ and $(x-1)$ are ≥ 0;

(ii) both $(x+2)$ and $(x-1)$ are ≤ 0;

(iii) $(x+2) \geq 0$ but $x-1 \leq 0$;

(iv) $(x+2) \leq 0$ but $x-1 \geq 0$.

Let X_1 consist of all x satisfying (i) and ($\sharp$). Show that $X_1 = (\frac{3}{2}, 2]$. Similarly $X_2 = \emptyset, X_3 = \emptyset$ and $X_4 = [-3, -\frac{5}{2})$.

Solution:

Method 1: Notice that

$$|x+2|+|x-1| = \begin{cases} 2x+1, & \text{if } x \geq 1, \\ 3, & \text{if} -2 \leq x \leq 1, \\ -2x-1, & \text{if } x \leq -2. \end{cases}$$

It follows that $X \cap [-2, 1) = \emptyset$. Consider $x \in [1, \infty) \cup (-\infty, -2)$.

If $x \in [1, \infty)$, then $4 < |x+2| + |x-1| < 5$ if and only if that $4 < 2x + 1 < 5$. This yields that $X \cap [1, \infty) = (\frac{3}{2}, 2]$.

If $x \in (-\infty, -2)$, then $4 < |x+2| + |x-1| < 5$ if and only if that $4 < -2x - 1 < 5$. From this inequality we have $X \cap (-\infty, -2) = [-3, -\frac{5}{2})$.

Therefore, we conclude that $X = [-3, -\frac{5}{2}) \cup (\frac{3}{2}, 2]$.

Method 2:

$$\begin{aligned} X \cap (-\infty, -2] &= \{x \in (-\infty, -2]; 4 < -2x - 1 \leqslant 5\} \\ &= \left\{x \in (-\infty, -2]; \frac{4-1\,|}{-2} > \frac{-2x-1+1}{-2} \geqslant \frac{5+1}{-2}\right\} \\ &= \left\{x \in (-\infty, -2] : -\frac{5}{2} > x \geqslant -3\right\} \\ &= \left[-3, -\frac{5}{2}\right); \end{aligned}$$

$$X \cap [-2, 1] = \{x \in [-2, 1]; \quad 4 < 3 \leqslant 5\} = \emptyset\};$$

$$\begin{aligned} X \cap [1, \infty) &= \{x \in [1, \infty) : \qquad 4 < 2x + 1 \leqslant 5\} \\ &= \{x \in [1, \infty) : \quad \frac{3}{2} < x \leqslant 2\} = \left(\frac{3}{2}, 2\right]. \end{aligned}$$

Therefore, since $\mathbb{R} = (-\infty, -2] \cup [-2, 1] \cup [1, \infty)$, one has

$$X = \left[-3, -\frac{5}{2}\right) \cup \left(\frac{3}{2}, 2\right].$$