Chapter 5: Plane Graphs

5.1 Surface Embeddings

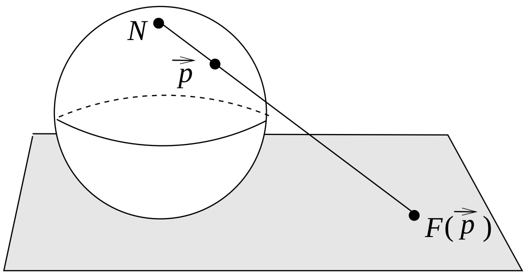
Definition 5.1.1: We say that G is *planar* if it can be drawn on the plane such that its edges intersect only at their end vertices. Such a drawing is called a *planar embedding* or *plane drawing.* A *plane graph* of G is a fixed embedding of G on the plane and a graph that is not planar is *nonplanar.*

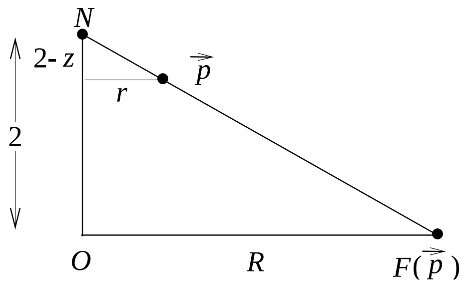
Remark 5.1.2: A planar graph must be isomorphic to a plane graph, and vice versa.

As a graph can be considered to embed into the plane, it can also be considered to embed into other surfaces, such as sphere, torus, double tours, Möbius band, etc. The definition can be generalized as follows:

Definition 5.1.3: Let G be a graph and S a surface. We say that G is *embeddable* on S if G is drawn in S with no crossing edges. Such a drawing is called an *embedding* of G on S.

Stereographic projection is a mapping F from the unit sphere S^2 that removed the north pole to the plane. Let S^2 be the unit sphere center at $(0, 0, 1)$ and let $N = (0, 0, 2)$ be the north pole. We imagine that there is a light source at the north pole, and a point $\overrightarrow{p}$ on $S^2 \setminus \{N\}$, let $F(\overrightarrow{p})$ be the shadow of $\overrightarrow{p}$.





Evidently F has the form

$$F(x, y, z) = \left(\frac{Rx}{r}, \frac{Ry}{r}\right),$$

where r and R are the distances from $\overrightarrow{p} = (x, y, z) \in S^2 \setminus \{N\}$ and $F(\overrightarrow{p})$ to the z-axis, respectively. It is easy to see that

$$F(x, y, z) = \left(\frac{2x}{2-z}, \frac{2y}{2-z}\right),$$

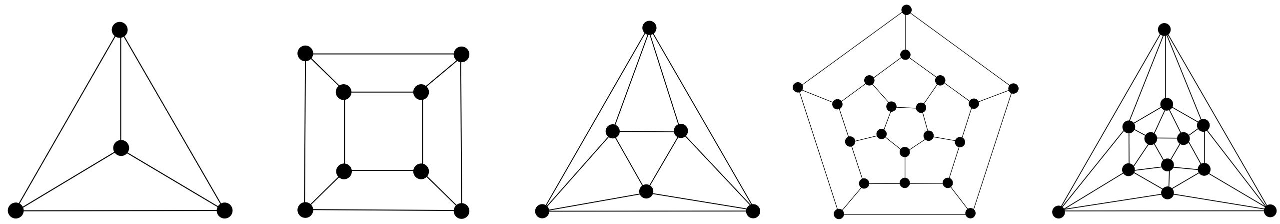
which is bijective and differentiable.

So we know that a sphere with one point removed and the plane have the same topological property (actually they are diffeomorphic). Apply the stereographic projection from sphere to the plane, we have

Theorem 5.1.4: *A graph is planar if and only if it is embeddable on the sphere.*

Note that if a graph is drawn in a sphere, then we may assume that the north point does not lie on the graph.

Thus the graphs formed by the lines and vertices of the five regular polyhedra are planar. Following are the plane drawings of these five regular polyhedra, which are called *Platonic graphs.* We shall prove that there are only five regular polyhedra.



Tetrahedron Hexahedron, Cube Octahedron Dodecahedron Icosahedron

It is clear that K_p is planar when $1 \le p \le 4$. We will prove that K_5 and $K_{3,3}$ are nonplanar. They are called the first and the second *Kuratowski graphs*, respectively.

Definition 5.1.5: For a plane graph G, there are regions $r_1, \ldots, r_\varphi$ in $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus G = r_1 \cup \cdots \cup r_\varphi$ and the following condition holds:

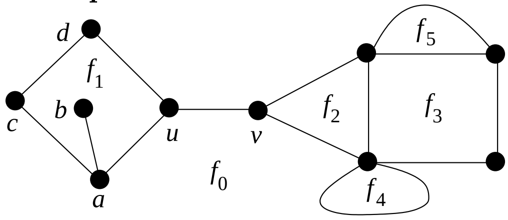
1. The sets $r_1, \ldots, r_\varphi$ are pairwise disjoint.
2. Each r_i is *path connected.* That is, for every two points $x, y \in r_i$, there is a continuous curve from x to y that does not intersect G.
3. Each continuous curve from $x \in r_i$ to $y \in r_j$, where $i \neq j$, intersects G at some point.

Each region r_i is called a *face* of the plane graph G. There is a unique unbounded face which is called the *outer face* of the plane graph G. Other faces are called *inner faces* of G.

Definition 5.1.6: Let f be a face of a plane graph G. We define the *degree* of f, denoted by $\deg(f)$ or $\deg_G(f)$, to be the number of edges on the boundary of f, where cut edges will be counted twice. Indeed, it is the smallest number of edges traversed among all closed walks around the boundary of f. If $\deg(f) = k$, then f is called a *k-face.* If all faces of G are k-faces, then G is called *face regular* or *k-face regular.*

Definition 5.1.7: Let f be a face of a plane graph G and $v \in V(G)$ that lies on the boundary of f. We say that v and f are *incident.* Similarly, an edge e and a face f are *incident* if e is on the boundary of f.

Example 5.1.8: Consider the following graph.



We have $\deg(f_1) = 6$, $\deg(f_2) = 3$, $\deg(f_3) = 4$, $\deg(f_4) = 1$, $\deg(f_5) = 2$, and $\deg(f_0) = 12$.

Note that ab belongs to the boundary of f_1. If we walks around the boundary of f_1 once, then we will travel ab twice and the shortest closed walk is $abacdua$, hence $\deg(f_1) = 6$.

Similarly we have $\deg(f_0) = 12$.

Given a plane graph G, we use $F(G)$ (or F) to denote the set of all faces of G. Sometimes we use $G = (V, E, F)$ to denote a plane graph and $\varphi(G)$ (or φ) to denote $|F|$. We have a lemma similar to the handshaking lemma.

Lemma 5.1.9: *Let $G = (V, E, F)$ be a (p, q)-plane graph. Then*

$$\sum_{f \in F} \deg(f) = 2q.$$

5.2 Euler's Formula

Theorem 5.2.1 (Euler's Formula)**:** *Every connected graph with p vertices, q edges and φ faces satisfies the equation $p - q + \varphi = 2$.*

Corollary 5.2.2: *Let G be a plane graph with p vertices, q edges, φ faces, and ω components. Then $p - q + \varphi = \omega + 1$.*

Corollary 5.2.3: *If G_1 and G_2 are two plane drawings of G, then $\varphi(G_1) = \varphi(G_2)$.*

Theorem 5.2.4: *Let G be a simple planar graph with p vertices and q edges, where $p \geq 3$. Then*

(a) $q \leq 3p - 6$.

(b) $q \leq 2p - 4$, if G has no 3-cycles.

Corollary 5.2.5: *K_5 is nonplanar.*

Corollary 5.2.6: *$K_{3,3}$ is nonplanar.*

Corollary 5.2.7: *Every simple connected planar graph whose minimum degree $\delta \leq 5$.*

Corollary 5.2.8: *There are only five regular polyhedra.*

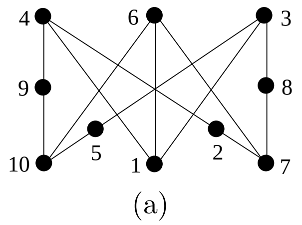
5.3 Planarity

If G is planar, then its subgraph is planar. The contrapositive of this statement is: If G contains a nonplanar subgraph, then G is nonplanar.

Definition 5.3.1: Let e be an edge. If we insert a vertex of degree two in e to make it become a path of length 2, then this operation is called a *subdivision* of e. A *subdivision* of a graph G is a graph obtained from G by a sequence of edge subdivisions.

Obviously, if G is planar, then any subdivision of G is planar. The converse also holds (see Theorem 5.3.8).

Example 5.3.2: Fig. 1(a) is a subdivision of $K_{3,3}$. It is isomorphic to a subgraph of the Petersen graph (Fig. 1(b)) and this shows that Petersen graph is nonplanar.



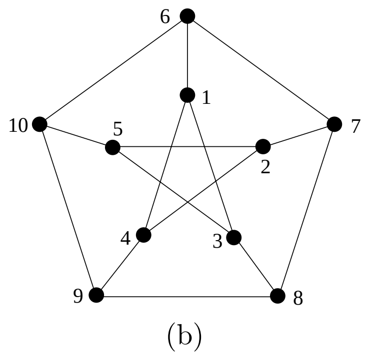


Figure 1: A subdivision of $K_{3,3}$ and Petersen graph.

Definition 5.3.5: A subgraph H of a graph that is a subdivision of K_5 or $K_{3,3}$ is called *Kuratowski subgraph* or *K-subgraph* for short.

Theorem 5.3.8 (Kuratowski, 1930)**:** *A graph is planar if and only if it does not contain K-subgraph.*

5.4 Duality

Given a planar graph G, a *plane dual* (or *dual graph*) G^* of G is defined by the following procedures:

1. Take a plane drawing $\widetilde{G}$ of G.

2. Choose one point r^* inside each face r of $\widetilde{G}$ — these points are the vertices of G^*.

3. For each e of $\widetilde{G}$, draw a line (curve) e^* connecting the vertices of G^* on each side of e. Note that if e is incident with the same face, then this line is a loop.

Moreover, suppose $V(G) = V(\widetilde{G}) = \{v_1, \ldots, v_p\}$, $E(G) = E(\widetilde{G}) = \{e_1, \ldots, e_q\}$ and $F(\widetilde{G}) = \{r_1, \ldots, r_\varphi\}$. Then $V(G^*) = \{r_1^*, \ldots, r_\varphi^*\}$ and $E(G^*) = \{e_1^*, \ldots, e_q^*\}$. It is easy to see that G^* is a connected plane graph. By Euler's formula, we have $\varphi(G^*) = p$.

More precisely, for a vertex $v \in V(G)$ of degree k, let $f_1, \ldots, f_k$ be the faces incident with v lying in the plane in clockwise direction. Note that they may not be different. Let a_i be the common edge incident with f_i and f_{i+1}, where $1 \le i \le k$ and $f_{k+1} = f_1$. The corresponding points for each face are $f_1^*, \ldots, f_k^*$. Then $f_1^* a_1^* f_2^* a_2^* \cdots f_k^* a_k^* f_1^*$ is a closed walk in G^* which around v in the plane. It indices a unique face containing v in G^* of degree k. We may denote this face by v^* and hence $F(G^*) = \{v_1^*, \ldots, v_p^*\}$.

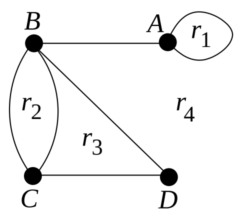
Using these corresponding, we have the following list:

an edge of G	$\leftrightarrow$	an edge of G^*
a vertex of degree k in G	$\leftrightarrow$	a face of degree k in G^*
a face of degree k in G	$\leftrightarrow$	a vertex of degree k in G^*

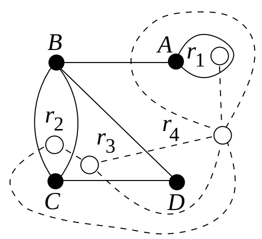
Note that the plane dual of G depends on the plane drawing of G. Difference plane drawing may induce non-isomorphic plane duals.

Example 5.4.1: Let $G = (V, E)$ be a graph with $V = \{A, B, C, D\}$, $E = \{AA, AB, BC, CB, BD, CD\}$.

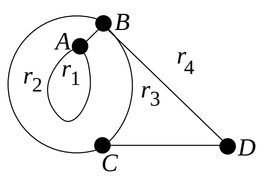
The following are two plane drawings of G and their corresponding plane duals (write vertices and dashed lines).



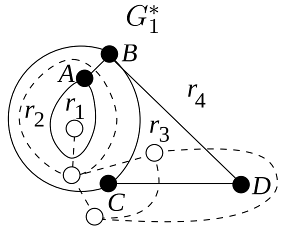
G_1



G_1^*



G_2



G_2^*

One can see that the plane duals are not isomorphic.

It can be proved that if G is a connected plane graph, then $(G^*)^* \cong G$. The proof is omitted here.