Chapter 1: Graphs and Networks

1.3 Notation and Basic Definitions

Definition 1.3.1: A *graph* is an ordered triple $G = (V, E, \phi)$, where $V \neq \varnothing$, $V \cap E = \varnothing$ and $\phi : E \to \mathscr{P}(V)$ such that $|\phi(e)| = 1$ or 2 for each $e \in E$.

Note that $\mathscr{P}(V)$ denotes the power set of V and the mapping ϕ is called the *end map* of the graph.

Definition 1.3.2: Elements of V are called *vertices* of G, and elements of E are called *edges* of G. The vertices in $\phi(e)$ are called the *end vertices* of the edge e.

Definition 1.3.6: Let $G = (V, E)$. When V and E are finite, the graph G is called a *finite graph*, and the cardinal of V is called the *order* of G and denoted by $|G|$ or $p(G)$, thus $|G| = |V|$. The number $|E|$ is called the *size* of G and sometimes denoted by $q(G)$. More precisely, if $|V| = p$ and $|E| = q$, then we say that G is a (p, q)*-graph*. A graph being not finite is called an *infinite graph*.

Note that, unless otherwise stated, the term "graph" always means finite graph.

Remark 1.3.7: *The graph G is also called an undirected graph. This is different from the digraph discussed later.*

Example 1.3.8: The graph come from Königsberg bridges problem is a $(4, 7)$-graph. That is, it contains 4 vertices and 7 edges.

Let G be a graph. We often use V and E to denote the vertex set and the edge set, respectively. When the discussion involves more than one graph, in order to avoid confusion we will use $V(G)$ and $E(G)$ to denote them, respectively.

We introduce some terminologies in the following:

Definition 1.3.9: Let $G = (V, E, \phi)$ be a graph.

1. An edge e is called a *loop* (resp. *link*) if $|\phi(e)| = 1$ (resp. $|\phi(e)| = 2$).
2. $u, v \in V$ are *adjacent* or *neighbors* if $\phi(e) = \{u, v\}$ for some $e \in E$. That is, $e = uv$ in our simplified notation. We also say that e joins u and v; u is adjacent with v and vice versa. Note that u and v may be the same.
3. $e, f \in E$ are *adjacent* if $\phi(e) \cap \phi(f) \neq \varnothing$, i.e., they have a common end vertex.
4. $u \in V$ and $e \in E$ are *incident* if $u \in \phi(e)$. Sometimes, we say that u is *incident with* e as well as e is *incident with* u.
5. $E' \subseteq E$ is a set of *multiple edges* or *parallel edges* if $|E'| \geq 2$ and $\phi(e) = \phi(f)$ for all $e, f \in E'$.
6. $u \in V$ is called *isolated* if $u \notin \phi(e)$ for all $e \in E$.

Definition 1.3.10: A graph containing no parallel edges nor no loops is called a *simple graph*. A graph that is not simple is called a *non-simple graph*. A graph containing parallel edges is called a *multigraph*.

1.4 Degree

Definition 1.4.1: Let $G = (V, E, \phi)$ be a graph and $u \in V$. The *degree* of u, denoted by $\deg_G(u)$, or $\deg(u)$ when there is no ambiguity, is defined by

$$\deg(u) = \Big|\{e \in E \mid u \in \phi(e),\ |\phi(e)| = 2\}\Big| + 2\Big|\{e \in E \mid u \in \phi(e),\ |\phi(e)| = 1\}\Big|.$$

That means $\deg(u)$ is the number of edges incident with u, where the loops are counted twice.

Theorem 1.4.3 (Handshaking Lemma, Euler)**:** *For a graph $G = (V, E)$, we have $\sum\limits_{u \in V} \deg(u) = 2|E|$.*

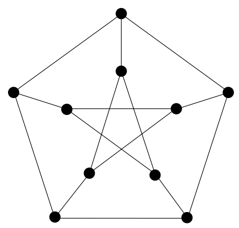
Definition 1.4.4: Let $G = (V, E)$ be a graph. A vertex u is called a *k-vertex* if $\deg(u) = k$. If k is odd (resp. even), then u is called an *odd vertex* (resp. *even vertex*).

Corollary 1.4.5: *In any graph there is always an even number of odd vertices.*

Corollary 1.4.6: *Suppose G is a (p, q)-simple graph (i.e., G is a (p, q)-graph and is also simple). Then $q \le \frac{1}{2}p(p-1)$.*

Definition 1.4.7: A graph $G = (V, E)$ is called *k-regular* if $\deg(u) = k$ for all $u \in V$. A 3-regular graph is also called a *cubic graph*. A graph is *regular* if it is k-regular for some nonnegative integer k.

The graph below is a famous cubic regular graph called the *Petersen graph*:



Corollary 1.4.8: *Every k-regular graph of order p has $\dfrac{kp}{2}$ edges.*

Definition 1.4.9: For a graph G with vertex set $V = \{v_1, \dots, v_p\}$, the sequence $(\deg(v_1), \dots, \deg(v_p))$ with $\deg(v_1) \ge \cdots \ge \deg(v_p)$ is called its *degree sequence*. The smallest term, $\deg(v_p)$, of the degree sequence is called the *minimum degree* of G and is denoted by $\delta(G)$ (or δ), while the largest term, $\deg(v_1)$, is called the *maximum degree* of G and is denoted by $\Delta(G)$ (or Δ).

From Handshaking Lemma we have

Corollary 1.4.10: *Suppose $(d_1, \dots, d_p)$ is a degree sequence of a graph, then $\sum\limits_{i=1}^{p} d_i$ is even.*

Definition 1.4.11: Let $G = (V, E)$ be a graph and $u \in V$. The *open neighborhood* of u (or *neighborhood* of u, for short), denoted by $N_G(u)$ or $N(u)$, is the set of all the neighbors of u in G. The *closed neighborhood* of u, denoted by $N_G[u]$ or $N[u]$, is defined by $N(u) \cup \{u\}$. In general, for $S \subseteq V$,

$$N_G(S) = \{v \mid v \in N_G(s) \text{ for some } s \in S\}$$
$$N_G[S] = N_G(S) \cup S.$$

When G is understood, we write $N(S)$ and $N[S]$, respectively.

Lemma 1.4.12: *Suppose $G = (V, E)$ is a graph and A as well as B are subsets of V. The following statements hold:*

1. $N(A \cup B) = N(A) \cup N(B)$;

2. $N[A \cup B] = N[A] \cup N[B]$;

3. $N(A \cap B) \subseteq N(A) \cap N(B)$;

4. $N[A \cap B] \subseteq N[A] \cap N[B]$.

1.5 Some Basic Graphs

The *null graph* on p vertices is the simple graph N_p, where $|V(N_p)| = p$ and $|E(N_p)| = 0$.

The *path graph* (or *path* for short) on $p \geq 2$ vertices is the simple graph P_p, where

$$V(P_p) = \{u_1, u_2, \ldots, u_p\} \text{ and}$$
$$E(P_p) = \{u_1u_2, u_2u_3, \ldots, u_{p-1}u_p\} = \{u_iu_{i+1} \mid 1 \leq i \leq p-1\}.$$

By convention, let $P_1 = N_1$. P_p is also called the *p-path.* It is easy to see that P_p is a $(p, p-1)$-graph.

The *cycle graph* (or simple *cycle*) of order $p \geq 3$ is the simple graph C_p, where

$$V(C_p) = \{u_1, u_2, \ldots, u_p\} \text{ and}$$
$$E(C_p) = \{u_1u_2, u_2u_3, \ldots, u_{p-1}u_p, u_pu_1\} = \{u_iu_{i+1} \mid 1 \leq i \leq p\} \text{ if we define } u_{p+1} = u_1.$$

Let $C_1 = (V_1, E_1)$, where $V_1 = \{u_1\}$, $E_1 = \{u_1u_1\}$.

Let $C_2 = (V_2, E_2)$, where $V_2 = \{u_1, u_2\}$, $E_2 = \{e_1 = u_1u_2, e_2 = u_1u_2\}$.

Thus C_p is simple if and only if $p \geq 3$. C_p is also called the *p-cycle.* C_3 is called a *triangle* and C_4 is called a *square.* Clearly, C_p is a (p, p)-graph.

A complete graph is a simple graph in which every two distinct vertices are adjacent. The *complete graph* of order p is denoted by K_p. Namely

$$V(K_p) = \{u_1, u_2, \ldots, u_p\} \text{ and}$$
$$E(K_p) = \{u_iu_j \mid 1 \leq i < j \leq p\}.$$

Proposition 1.5.1: *The complete graph K_p is a $(p-1)$-regular graph and contains $\frac{1}{2}p(p-1)$ edges.*

Proof: Every vertex is adjacent to other $p-1$ vertices, so K_p is $(p-1)$-regular. By Corollary 1.4.8, the number of edges in K_p is $\frac{1}{2}p(p-1)$. □

Proposition 1.5.2: *Suppose G is a (p, q)-simple graph. Then $q = \frac{1}{2}[p(p-1)]$ if and only if G is K_p.*

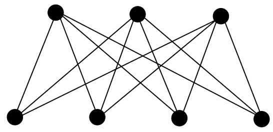
Suppose $k \in \mathbb{N}$. A graph $G = (V, E)$ is said to be *k-partite* if V can be partitioned into k disjoint subsets $V_1, \ldots, V_k$, such that no two vertices within the same set are adjacent. The partition $(V_1, \ldots, V_k)$ is called a *k-partition* of G. When $k = 2$, G is called *bipartite* and (V_1, V_2) is a *bipartition* of G.

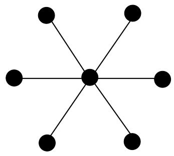
A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which every vertex of X is adjacent to all vertices of Y. If $|X| = m$ and $|Y| = n$, then such a graph is denoted by $K_{m,n}$. The vertex set and edge set of $K_{m,n}$ are

$$V(K_{m,n}) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\} \text{ and}$$
$$E(K_{m,n}) = \{u_iv_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

It is easy to see that $K_{m,n}$ and $K_{n,m}$ are the same in some sense (it will be called isomorphic). Hence we can assume $m \leq n$.

The complete bipartite graph $K_{1,n}$ is a called a *star* (or *n-star*) and denoted by S_n.





The following proposition can be obtained easily.

Proposition 1.5.3: *The complete bipartite graph $K_{m,n}$ is an $(m+n, mn)$-graph.*

Proposition 1.5.4: *Suppose G is a simple bipartite graph with p vertices and q edges, then $q \leq \frac{p^2}{4}$.*

1.6 Subgraphs

Definition 1.6.1: For graphs $G' = (V', E')$ and $G = (V, E)$. G' is called a *subgraph* of G (also G is a *supergraph* of G') if $V' \subseteq V$ and $E' \subseteq E$. We write $G' \subseteq G$ if G' is a subgraph of G. When $G' \subseteq G$ and $G' \neq G$, G' is a *proper subgraph* of G and is denoted as $G' \subset G$.

It is easy to obtain the following properties about subgraphs:

(1) *$G \subseteq G$.*

(2) *If $F \subseteq H$ and $H \subseteq G$, then $F \subseteq G$.*

(3) *Suppose $G = (V, E)$. For each $v \in V$, $H = (\{v\}, \varnothing)$ is a subgraph of G.*

(4) *A graph obtained from some edges of G together with their end vertices is a subgraph of G. Such a subgraph is called an edge-induced subgraph of G. We will give a formal definition later.*

Definition 1.6.3: For a nonempty subset $W \subseteq V(G)$, the subgraph of G *induced by W*, denoted by $G[W]$, is the graph with vertex set W whose edge set consists of all the edges of G having their end vertices in W. $G[W]$ is also called the *induced subgraph* of G by W.

In other words, $G[W]$ is the maximal (with respect to inclusion '$\subseteq$') subgraph of G containing W.

Definition 1.6.4: For a nonempty subset $F \subseteq E(G)$, the subgraph of G *induced by F*, denoted by $G[F]$, is the graph with edge set F whose vertex set consists of all the end vertices of F. $G[F]$ is also called the *edge-induced subgraph* of G by F.

In other words, $G[F]$ is the minimal subgraph of G containing F.

When $W = \{w_1, \ldots, w_m\} \subseteq V(G)$. We write $G[w_1, \ldots, w_m]$ instead of $G[\{w_1, \ldots, w_m\}]$ and similar convention will be adopted for the edge-induced subgraph.

Definition 1.6.5: Let $H \subseteq G$. If $V(H) = V(G)$, then H is called a *spanning subgraph* of G. A subgraph obtained from G by deleting all loops and identifying all parallel edges is called the *basic simple graph* or *underlying simple graph* of G.

Thus the underlying simple graph must be a spanning subgraph of G.

1.7 Isomorphism

Definition 1.7.1: Let $G = (V, E, \phi)$ and $H = (V', E', \psi)$ be two graphs. An *isomorphism* f from G to H is an ordered pair $f = (f_V, f_E)$ of bijections $f_V : V \to V'$ and $f_E : E \to E'$ satisfying the following condition:

$$\phi(e) = \{u, v\} \text{ implies } \psi(f_E(e)) = \{f_V(u), f_V(v)\}.$$

That is, f preserves the adjacency of vertices.

Two graphs G and H are *isomorphic*, denoted by $G \cong H$, if there is an isomorphism between them.

For simple graphs, we may ignore the end maps.

Definition 1.7.2: Let $G = (V, E)$ and $H = (V', E')$ be two simple graphs. An *isomorphism* $f : G \to H$ is an ordered pair $f = (f_V, f_E)$ of bijections $f_V : V \to V'$ and $f_E : E \to E'$ satisfying the following condition:

$$f_E(uv) = f_V(u)f_V(v)$$

for every edge $uv \in E$.

Definition 1.7.3: Two graphs G and H are *equal* or *identical*, denoted by $G = H$, if $V(G) = V(H)$, $E(G) = E(H)$ and their end maps are the same.

Remark 1.7.4: *Of course, two identical graphs are isomorphic. We may treat isomorphic graphs as the same graphs.*

Definition 1.7.6: Any subgraph of a given graph G that is isomorphic to a complete graph K_h for some $h \in \mathbb{N}$ is called an *h-clique* in G. When the number of vertices in the subgraph is irrelevant, we simply call it a *clique*.

Suppose $f : G \to H$ is an isomorphism. Then

1. $|V(G)| = |V(H)|$.
2. $|E(G)| = |E(H)|$.
3. $\deg_G(u) = \deg_H(f_V(u))$.
4. The degree sequence of G is the same as that of H.
5. The number of h-cliques in G is the same as that in H.
6. $\vdots$

Definition 1.7.8: For simple graph $G = (V, E)$, the *complement* of G is defined as $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{uv \mid u, v \in V,\ uv \notin E\}$.

Clearly $\overline{N}_n = K_n$, $\overline{K}_n = N_n$, $\overline{\overline{G}} = G$. $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

1.8 Graph Operations and Constructions

Definition 1.8.1: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs.

1. The *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

2. If $V_1 \cap V_2 = \varnothing$, then G_1 and G_2 are *disjoint*. If $E_1 \cap E_2 = \varnothing$, then G_1 and G_2 are *edge-disjoint*.

3. If G_1 and G_2 are disjoint, then the union $G_1 \cup G_2$ is called the *disjoint union* of G_1 and G_2, and denoted by $G_1 + G_2$.

4. If $V_1 \cap V_2 \neq \varnothing$, then the graph $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ is called the *intersection* of G_1 and G_2.

5. The *symmetric difference* of G_1 and G_2 is the graph $G_1 \triangle G_2 = (V_1 \cup V_2, E_1 \triangle E_2)$.

Note that all set operations related to edge sets are multiset operations. It is easy to see that operations $\cup$, $+$, $\cap$ and $\triangle$ satisfy association and symmetric laws.

Definition 1.8.2: The *join* of two disjoint graphs G and H, denoted by $G \vee H$, is obtained from $G + H$ by joining each vertex of G to all vertices of H.

Clearly $G \vee H \cong H \vee G$ and $(G_1 \vee G_2) \vee G_3 \cong G_1 \vee (G_2 \vee G_3)$.

Definition 1.8.3: The *sequential join* $G_1 \vee G_2 \vee \cdots \vee G_k$ of graphs $G_1, G_2, \ldots, G_k$ is the graph formed by taking one copy of each graph and adding additional edges from each vertex of G_i to all vertices of G_{i+1}, for $1 \leq i \leq k-1$.

Note that $G_1 \vee G_2 \vee G_3 \ncong (G_1 \vee G_2) \vee G_3$.

Clearly $K_{m,n} \cong N_m \vee N_n$. For $n \geq 3$, the *wheel graph* W_{n+1} is defined by $W_{n+1} = K_1 \vee C_n$. For $n \geq 2$, the *fan graph* F_{n+1} is defined by $F_{n+1} = K_1 \vee P_n$.

Definition 1.8.4: For graphs G and H, the *Cartesian product* $G \times H$ (or some books use $G \Box H$) has vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either

1. $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or

2. $v_1 = v_2$ and $u_1 u_2 \in E(G)$.

We call an edge of the first type an *H edge*, and that of the second type a *G edge*.

It is clear that $G \times H \cong H \times G$ and $(G \times H) \times K \cong G \times (H \times K)$. Thus we may denote this product as $G \times H \times K$. There is a similar definition for the Cartesian product of more graphs.

The *hypercube* Q_n or (*n-cube*) is defined recursively: $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. That is, $Q_n = \overbrace{K_2 \times \cdots \times K_2}^{n \text{ times}}$.

Each vertex of Q_n can be labeled as a binary sequence of length n, or equivalently $V(Q_n) = \mathbb{Z}_2^n$. Two vertices in Q_n are adjacent if and only if their coordinates differ in exactly one place.

$P_m \times P_n$ is called a *mesh* or a *grid*. It is also called a 2-*mesh* and denoted by $M(m, n)$. The *n-mesh* $M(a_1, \ldots, a_n)$ is the Cartesian product $P_{a_1} \times \cdots \times P_{a_n}$.

Definition 1.8.5: For any graph G, its *line graph* $L(G)$ has vertex set consisting of the edges of G, i.e., $V(L(G)) = E(G)$. Two vertices of $L(G)$ are adjacent if the corresponding edges of G have a vertex in common.

Definition 1.8.6: If $G = (V, E)$, for $U \subset V$, the subgraph $G - U$ is obtained from G by removing all the vertices in U and all the edges of G incident with vertices in U. That is, $G - U = G[V \setminus U]$.

If $U = \{v_1, \ldots, v_s\}$, then $G - U$ is often written as $G - v_1 - \cdots - v_s$.

Definition 1.8.7: If $G = (V, E)$, for $F \subseteq E$, the subgraph $G - F$ is obtained from G by removing all the edges in F. Note that all vertices of G are retained.

If $F = \{e_1, \dots, e_t\}$, then $G - F$ is often written as $G - e_1 - \cdots - e_t$.

Definition 1.8.8: Let G be a graph. Suppose E' is a set of edges which are not in G but their end vertices are vertices of G. $G + E'$ denotes the graph obtained from G by adding all edges of E' to G. If $E' = \{e\}$, then we simply denote $G + E'$ by $G + e$.

A non-increasing sequence S of nonnegative integers is called *graphical* if there is a simple graph whose degree sequence is S.

Theorem 1.8.9: *If a sequence of nonnegative integers $(d_1, \dots, d_p)$ with $d_1 \geq \cdots \geq d_p$ is graphical, then $\sum\limits_{i=1}^{p} d_i$ is even and for each integer k $(1 \leq k < p)$*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{p} \min\{k, d_j\}.$$

Theorem 1.8.10: *Suppose $S = (d_1, d_2, \dots, d_p)$ is a non-increasing sequence of nonnegative integers. Let*

$$S' = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_p),$$

where $d_1 + 1 \leq p$. Let S^* be the non-increasing sequence obtained from S' by rearranging the terms of S'. Then S is graphical if and only if S^* is graphical.

Algorithm: Determining Graphical Degree Sequence

Given a non-increasing sequence S of nonnegative integers.

Step 1. Delete the first number, say k, from S.

Step 2. Subtract 1 from each of the next k terms of S if this is possible. The resulting sequence is denoted by S'. If S' cannot be formed, stop; the original sequence is not graphical. If all terms of the current sequence are zero, stop; the original sequence is graphical.

Step 3. Rearrange the sequence obtained so that it is a sequence S^* in non-increasing order.

Step 4. Let $S = S^*$, and return to Step 1.

1.9 Directed Graph

Definition 1.9.1: A *directed graph* or *digraph* or *network* is an ordered triple $\overrightarrow{G} = (V, E, \eta)$, where $V \neq \varnothing$, $V \cap E = \varnothing$ and $\eta : E \to V \times V$ is a map.

Elements of V are called *vertices* of G, and elements of E are called *directed edges* or *arcs*. If $\eta(e) = (u, v)$, then u is called the *tail* of e and v the *head* of e. Vertex u is called a *predecessor* of v and v is called a *successor* of u. We say that u is *adjacent to* v while v is *adjacent from* u. Also, u is *incident to* e and v is *incident from* e. A vertex not incident with any arc is called an *isolated vertex*.

If $\eta(e) = (u, u)$, then e is called a *directed loop*. The vertex u is a tail of e and also a head of e.

Two arcs e and f are said to be *parallel* if $\eta(e) = \eta(f)$.

We will assume that all digraphs are finite.

Definition 1.9.2: Given a digraph, the graph with each arc replaced by an edge is called the *underlying graph*. That is, if $\overrightarrow{G} = (V, E, \eta)$ and the corresponding underlying graph is $G = (V, E, \phi)$, then $\eta(e) = (u, v) \Rightarrow \phi(e) = \{u, v\}$.

Definition 1.9.3: A digraph without directed loops and parallel arcs is called a *simple digraph*.

Similar to undirected graph, we use V and E to denote the vertex set and arc set of a digraph $\overrightarrow{G}$, respectively. When the discussion involves more than one digraph, in order to avoid confusion, we use $V(\overrightarrow{G})$ and $E(\overrightarrow{G})$ to denote them, respectively. We use (u, v) to denote an arc, where u is the tail and v is the head of the arc. When we adopt this notation, the end map η may be omitted. $\overrightarrow{G}$ is also called a (p, q)-*digraph* if $\overrightarrow{G}$ contains p vertices and q arcs. Moreover, we use $p(\overrightarrow{G})$ and $q(\overrightarrow{G})$ to denote the order and the size of $\overrightarrow{G}$.

The *directed path* on $p \geq 2$ vertices is the simple digraph $\overrightarrow{P}_p$ with

$$\begin{aligned} V(\overrightarrow{P}_p) &= \{u_1, \ldots, u_p\} \\ E(\overrightarrow{P}_p) &= \{(u_1, u_2), \ldots, (u_{p-1}, u_p)\} = \{(u_i, u_{i+1}) \mid 1 \leq i \leq p-1\} \end{aligned}$$

The *directed cycle* on $p \geq 2$ vertices is the simple digraph $\overrightarrow{C}_p$ with

$$\begin{aligned} V(\overrightarrow{C}_p) &= \{u_1, \ldots, u_p\} \\ E(\overrightarrow{C}_p) &= \{(u_1, u_2), \ldots, (u_{p-1}, u_p), (u_p, u_1)\} = \{(u_i, u_{i+1}) \mid 1 \leq i \leq p\}, \text{ where } u_{p+1} = u_1. \end{aligned}$$

$\overrightarrow{C}_1$ is defined to be the digraph consisting of a directed loop.

Definition 1.9.6: Let $\overrightarrow{G} = (V, E, \eta)$ be a digraph. Let $u \in V$.

1. The *indegree* of u, denoted by $\deg^-_{\overrightarrow{G}}(u)$ or $\deg^-(u)$, is the number of arcs having u as head.
2. The *outdegree* of u, denoted by $\deg^+_{\overrightarrow{G}}(u)$ or $\deg^+(u)$, is the number of arcs having u as tail.
3. The *inneighborhood*, denoted by $N^-_{\overrightarrow{G}}(u)$ or $N^-(u)$, and the *outneighborhood*, denoted by $N^+_{\overrightarrow{G}}(u)$ or $N^+(u)$ of u are given by

$$\begin{aligned} N^-(u) &= \{x \in V \mid \eta(e) = (x, u) \text{ for some } e \in E\}, \\ N^+(u) &= \{x \in V \mid \eta(e) = (u, x) \text{ for some } e \in E\}. \end{aligned}$$

Theorem 1.9.7: **(Handshaking Lemma)** *For a digraph $\overrightarrow{G}$, we have*

$$\sum_{u \in V(\overrightarrow{G})} \deg^-(u) = \sum_{u \in V(\overrightarrow{G})} \deg^+(u) = |E(\overrightarrow{G})|.$$

Definition 1.9.8: A digraph $\overrightarrow{G}$ is called *balanced* if for every vertex u of $\overrightarrow{G}$, $\deg^-(u) = \deg^+(u)$. A balanced digraph G is called *regular* if $\deg^-(u) = \deg^-(v)$ for all $u, v \in V(\overrightarrow{G})$.

1.10 Matrix Representations

Definition 1.10.1: Let $G = (V, E)$ be a graph. Suppose we give each vertex of G a name (label). Then the graph G is called a *labeled graph*. Mathematically, there is a bijection $f : V \to L$, where L is a set. The bijection f is called a *labeling* (or *vertex labeling*) of G and L is called a *label set*.

For general labeling problems, the labeling may not be bijective.

Similarly, we may label edges of a graph and such a labeling is called an *edge labeling*. We omit the formal definition here.

Definition 1.10.2: Let $G = (V, E, \phi)$ be a graph on p vertices labeled by $\{u_1, \dots, u_p\}$. For each $i, j \in \{1, \dots, p\}$ define

$$a_{ij} = |\{e \in E \mid \phi(e) = \{u_i, u_j\}\}|.$$

That is, a_{ij} is the number of edges connecting u_i and u_j. The *adjacency matrix* of G with respect to the labeling is defined as the $p \times p$ matrix $A(G) = (a_{ij})$.

Remark 1.10.3: *Let $G = (V, E)$ be a graph.*

1. *Clearly, $A(G)$ is symmetric.*
2. *If G is a simple graph, then $A(G)$ is a symmetric binary matrix. That is, it is symmetric and $a_{ij} \in \{0, 1\}$ for all i and j. In addition, $a_{ii} = 0$ for all i.*
3. $|E(G)| = \sum\limits_{i \geq j} a_{ij} = \sum\limits_{i \leq j} a_{ij}$.
4. $\deg(u_i) = a_{ii} + \sum\limits_{\ell=1}^{n} a_{i\ell} = a_{ii} + \sum\limits_{\ell=1}^{n} a_{\ell\, i}$.

Let A be an adjacency matrix of G with respect to a labeling. If we relabel all the vertices, then the adjacency matrix of G, say A', with respect to the new labeling is obtained by permuting the rows and the columns of A accordingly. In other words, $A' = PAP^T$, for some permutation matrix P, where P^T is the transpose of P. In this case, $P^T = P^{-1}$.

Since two graphs being isomorphic means that the vertices of one graph can be rearranged to match the other. Therefore, by the discussion above, we obtain the following proposition.

Proposition 1.10.4: *Graphs G and H are isomorphic if and only if there is a permutation matrix P such that $A(H) = P^{-1}A(G)P$.*

Corollary 1.10.5: *If $G \cong H$, then the spectrum of their adjacency matrices are the same, i.e., the multisets of their eigenvalues are the same.*

Proposition 1.10.8: *Let G be a bipartite graph with bipartition (X, Y). If $|X| = m$ and $|Y| = n$, then an adjacency matrix of G is of the form*

$$\left(\begin{array}{c|c} O_m & B \\ \hline B^T & O_n \end{array}\right),$$

for some matrix B, where O_k is the square zero matrix of order k.

Let J_p be the $p \times p$ matrix with all entries equal to 1 and I_p be the identity matrix of order p. It is easy to obtain the following proposition.

Proposition 1.10.9: *The adjacency matrix of K_p is $J_p - I_p$. If G is a simple graph of order p, then the adjacency matrix of $\overline{G}$ is $J_p - I_p - A(G)$.*

Adjacency matrix describes the relationship between vertices. Another matrix called incident matrix describes the relationship between vertices and edges.

Definition 1.10.10: Let $G = (V, E)$ be a graph with vertex labeling $V = \{u_1, \ldots, u_p\}$ and edge labeling $E = \{e_1, \ldots, e_q\}$. The *incidence matrix* of G with respect to these labelings is the $p \times q$ matrix $M(G) = (m_{ij})$, where the (i, j)-th entry m_{ij} is defined by

$$m_{ij} = \begin{cases} 0 & \text{if } u_i \text{ is not incidence with } e_j, \\ 1 & \text{if } u_i \text{ is incidence with a link } e_j, \\ 2 & \text{if } u_i \text{ is incidence with a loop } e_j. \end{cases}$$

Remark 1.10.11: *By the definition of $M(G)$, the i-th row sum is equal to the degree of u_i, while the sum of each column is 2. Using this property we can prove the handshaking lemma again.*

Theorem 1.10.13: *Let $G = (V, E)$ be a simple graph. Suppose $V = \{u_1, \ldots, u_p\}$ and $E = \{e_1, \ldots, e_q\}$. Let A and M be the adjacency and incidence matrices of G, respectively. Then $MM^T = A + D$, where $D = \mathrm{diag}\{\deg(u_1), \ldots, \deg(u_p)\}$.*

Definition 1.10.14: Let $\overrightarrow{G}$ be a digraph of order p. We label the vertices of G as $v_1, \ldots, v_p$. The *adjacency matrix* of $\overrightarrow{G}$, with respect to this labeling, is the $p \times p$ matrix $A(\overrightarrow{G}) = (a_{ij})$, where the (i, j)-th entry a_{ij} is the number of arcs joining from v_i to v_j.

Note that $A(\overrightarrow{G})$ may not be symmetric. Moreover, $|E(\overrightarrow{G})| = \sum\limits_{i,j} a_{ij}$, the i-th row sum is $\deg^+(v_i)$ and the i-th column sum is the $\deg^-(v_i)$.

Definition 1.10.16: Let $\overrightarrow{G}$ be a (p, q)-digraph without loop. Label the vertices as $v_1, \ldots, v_p$ and the arcs $a_1, \ldots, a_q$. The *incidence matrix* of $\overrightarrow{G}$, with respect to these labelings, is the $p \times q$ matrix $M(\overrightarrow{G}) = (m_{ij})$, where the (i, j)-th entry m_{ij} is defined by

$$m_{ij} = \begin{cases} 1 & \text{if the arc } a_j \text{ is incidence from the vertex } v_i, \\ -1 & \text{if the arc } a_j \text{ is incidence to the vertex } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the i-th row sum is $\deg^+(v_i) - \deg^-(v_i)$, while the sum of each column is 0.