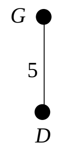
MMAT5380 Graph Theory and Networks
Suggested Solution for Assignment 4

4-1: Let (X, Y) be bipartition of G. Let $W = u_1u_2 \cdots u_ku_1$ be a closed walk of length k. Without loss of generality, we may assume that $u_1 \in X$. Then $u_2 \in Y$, $u_3 \in X$ and so on. In general, $u_i \in X$ for odd i and $u_j \in Y$ for even j. Since $u_1 \in X$, $u_k \in Y$. Hence k must be even.

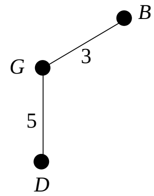
4-2: (a) The chosen list of edges are $BC(2)$, $AB(3)$, $BG(3)$, $DG(5)$, $AF(6)$, $EF(7)$.

(b)

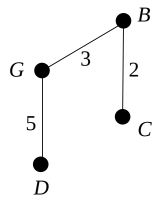
	A	B	C	$\underline{D}$	E	F	G
A	$*$	3	∞	∞	∞	6	4
B	3	$*$	2	∞	∞	∞	3
C	∞	2	$*$	10	∞	∞	6
E	∞	∞	∞	10	$*$	7	9
F	6	∞	∞	∞	7	$*$	8
G	4	3	6	$\boxed{5}$	9	8	$*$



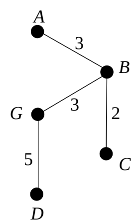
	A	B	C	$\underline{D}$	E	F	$\underline{G}$
A	$*$	3	∞	∞	∞	6	4
B	3	$*$	2	∞	∞	∞	$\boxed{3}$
C	∞	2	$*$	10	∞	∞	6
E	∞	∞	∞	10	$*$	7	9
F	6	∞	∞	∞	7	$*$	8



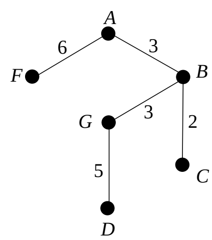
	A	$\underline{B}$	C	$\underline{D}$	E	F	$\underline{G}$
A	$*$	3	∞	∞	∞	6	4
C	∞	$\boxed{2}$	$*$	10	∞	∞	6
E	∞	∞	∞	10	$*$	7	9
F	6	∞	∞	∞	7	$*$	8



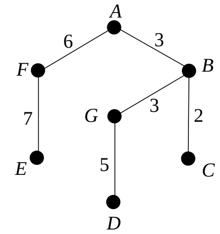
	A	$\underline{B}$	$\underline{C}$	$\underline{D}$	E	F	$\underline{G}$
A	$*$	$\boxed{3}$	∞	∞	∞	6	4
E	∞	∞	∞	10	$*$	7	9
F	6	∞	∞	∞	7	$*$	8



	$\underline{A}$	$\underline{B}$	$\underline{C}$	$\underline{D}$	E	F	$\underline{G}$
E	∞	∞	∞	10	$*$	7	9
F	$\boxed{6}$	∞	∞	∞	7	$*$	8



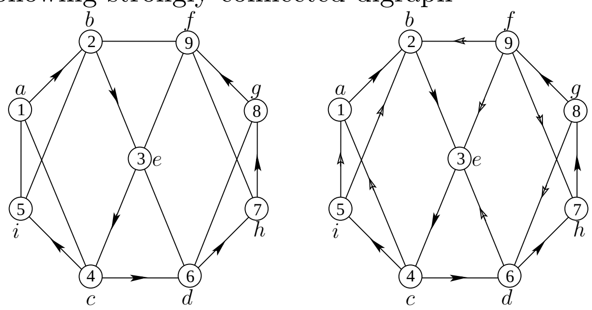
	$\underline{A}$	$\underline{B}$	$\underline{C}$	$\underline{D}$	E	$\underline{F}$	$\underline{G}$
E	∞	∞	∞	10	$*$	$\boxed{7}$	9



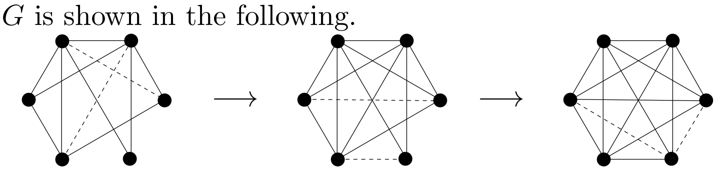
4-3: (a) Use a, b, c, e, i, h, g, d, f as the pre-ordering and we take a as the root.

Pass	Step	$N(b^*) \cap U$	w	$\overset{R}{\uparrow}$ b^*w	$l(w)$	U	b^*	i
0	1			$\varnothing$	$(1, *)$	b, c, e, i, h, g, d, f	a	2
1	2	b, c, i	b	ab	$(2, a)$	c, e, i, h, g, d, f	b	3
2	2	e, i, f	e	be	$(3, b)$	c, i, h, g, d, f	e	4
3	2	c, d, f	c	ec	$(4, e)$	i, h, g, d, f	c	5
4	2	i, d	i	ci	$(5, c)$	h, g, d, f	i	6
5	$2\#, 3$	$\varnothing$					c	
6	2	d	d	cd	$(6, c)$	h, g, f	d	7
7	2	h, g	h	dh	$(7, d)$	g, f	h	8
8	2	g, f	g	hg	$(8, h)$	f	g	9
9	2	f	f	gf	$(9, g)$	$\varnothing$		

(b) We obtain the following strongly connected digraph



4-4: The closure of G is shown in the following.



4-5: View each $1 \times 1 \times 1$ cube as a vertex of a graph. Two vertices are adjacent if the corresponding cubes are adjacent. Then the corresponding graphs is isomorphic to $P_3 \times P_3 \times P_3$. It is clearly that it is bipartite.

So we can color the vertices by white or black such that each pair of adjacent vertices have differ colors. Suppose we color all vertices corresponding corners by black, then the vertex corresponding the center will be colored by white.

Suppose the mouse starts at a corner and it can finish at the center. That means that the corresponding graph has a Hamiltonian path from a corner vertex (say v) to the center vertex (say c). If we add an edge vc on the graph, then the resulting graph has a Hamiltonian cycle. It is impossible, since the graph is a bipartite graph with odd order.

4-6: (a) C_p with $p \geq 5$.

(b) $K_{n-1,n}$ for $n \geq 2$, where $p = 2n - 1$. (If $K_{n-1,n}$ contains a Hamiltonian cycle, then it must be an odd cycle. But bipartite graph does not contain any odd cycle.)

4-7: Let X be an independent set of G. We take $S = N(X)$. Since X is an independent set, after deleting $N(X)$ it creates at least $|X|$ components which are isolated vertices. Thus $\omega(G - N(X)) \geq |X|$. From the hypothesis $|X| > |N(X)|$, we have $\omega(G - N(X)) > |N(X)|$. By Theorem 4.2.3, G is non-hamiltonian.

4-8: Let $H = G \vee K_1$. Since $\deg_G(u) + \deg_G(v) \geq |G| - 1$ for any two nonadjacent vertices u, v of G, we have $\deg_H(u) + \deg_H(v) \geq |G| + 1 = |H|$ for any two nonadjacent vertices u, v of H. Then by Corollary 4.2.9, H is Hamiltonian. Let C be a Hamiltonian cycle of H. Then, obviously, $C - x$ is a Hamiltonian path of G.