1. **Definition. (Systems of linear equations.)**

Let a_{ij} *be (fixed) real numbers for each* $i = 1, \cdots, m$ *and for each* $j = 1, \cdots, n$.

Let b_k *be (fixed) real numbers for each* $k = 1, \cdots, m$.

(a) *The system of* m *simultaneous equations with unknowns* $x_1, x_2, \cdots, x_n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

is called a system of m *linear equations with* n *unknowns.*

The numbers a_{ij}*'s,* b_k*'s are referred to as givens in this system of linear equations.*

(b) *Denote such a system of linear equations by* (S).

Let $t_1, t_2, \cdots, t_n$ *be (fixed) real numbers.*

We say $(x_1, x_2, \cdots, x_n) = (t_1, t_2, \cdots, t_n)$ *is a solution of the system* (S) *if and only if the* m *equalities*

$$\begin{array}{ccccccccc} a_{11}t_1 & + & a_{12}t_2 & + & \cdots & + & a_{1n}t_n & = & b_1, \\ a_{21}t_1 & + & a_{22}t_2 & + & \cdots & + & a_{2n}t_n & = & b_2, \\ & & & & & & & \vdots & \\ a_{m1}t_1 & + & a_{m2}t_2 & + & \cdots & + & a_{mn}t_n & = & b_m \end{array}$$

hold simultaneously.

(c) *(Again denote such a system of linear equations by* (S)*.)*

i. *We say* (S) *is consistent if and only if there is some solution for* (S).

ii. *We say* (S) *is inconsistent if and only if there is no solution for* (S).

2. **Definition. (Equation operation 'adding a scalar multiple of one equation to another'.)**

Consider the system of linear equations

$$(S): \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

in which $x_1, x_2, \cdots, x_n$ are the unknowns.

Suppose α is a real number.

When we replace the k-th equation

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

of (S) by the equation

$$(\alpha a_{i1} + a_{k1})x_1 + (\alpha a_{i2} + a_{k2})x_2 + \cdots + (\alpha a_{in} + a_{kn})x_n = \alpha b_i + b_k,$$

in which $i \neq k$, to obtain some (other) system, we say we are applying the equation operation '$\alpha \times ⓘ + ⓚ$' to (S).

Such an equation operation is called 'adding a scalar multiple of one equation of (S) to another equation of (S)'.

3. **Definition. (Equation operation 'multiplying a non-zero scalar to one equation'.)**

Consider the system of linear equations

$$(S):\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

in which $x_1, x_2, \cdots, x_n$ *are the unknowns.*

Suppose β *is a non-zero real number.*

When we replace the k*-th equation*

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

of (S) *by the equation*

$$\beta a_{k1}x_1 + \beta a_{k2}x_2 + \cdots + \beta a_{kn}x_n = \beta b_k,$$

to obtain some (other) system, we say we are applying the equation operation '$\beta \times \textcircled{k}$' *to* (S).

Such an equation operation is called 'multiplying a non-zero scalar to one equation of (S)'.

4. **Definition. (Equation operation 'interchanging two equations'.)**

Consider the system of linear equations

$$(S) : \begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

in which $x_1, x_2, \cdots, x_n$ *are the unknowns.*

When we interchange the i*-th equation*

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

of (S) *and the* k*-th equation*

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

of (S)*, in which* $i \neq k$*, to obtain some (other) system, we say we are applying the equation operation '*$\textcircled{i} \leftrightarrow \textcircled{k}$*' to* (S)*.*

Such an equation operation is called 'interchanging two equations of (S)*'.*

5. **Definition. (Equation operations.)**

Let $(S), (T)$ be systems of m linear equations with n unknowns.

We say we are applying one equation operation on (S) to obtain the system (T) if and only if (T) is the resultant of the application of

- *one equation operation 'adding a scalar multiple of one equation of (S) to another', or*
- *one equation operation 'multiplying a non-zero scalar to one equation (S)', or*
- *one equation operation 'interchanging two equations of (S)'.*

6. **Definition. (Equivalent systems of linear equations.)**

Let $(S), (T)$ be systems of m linear equations with n unknowns.

We say (S) is equivalent to (T) as systems if and only if both statements below hold:

(a) *Every solution of (S) is a solution of (T).*

(b) *Every solution of (T) is a solution of (S).*

7. **Theorem (1).**

Let $(S), (T)$ be systems of m linear equations with n unknowns.

(a) *Suppose (T) is resultant from the application of one equation operation on (S). Then (S) is equivalent to (T) as systems.*

(b) *Suppose (T) is resultant from the application of finitely many equation operations, starting from (S). Then (S) is equivalent to (T) as systems.*

8. **Theorem (2).**

The statements below hold:

(a) *Suppose (S) is a system of m linear equations with n unknowns. Then (S) is equivalent to (S) as systems.*

(b) *Let $(S), (T)$ be systems of m linear equations with n unknowns.*
Suppose (S) is equivalent to (T) as systems. Then (T) is equivalent to (S) as systems.

(c) *Let $(S), (T), (U)$ be systems of m linear equations with n unknowns.*
Suppose (S) is equivalent to (T) as systems, and (T) is equivalent to (U) as systems. Then (S) is equivalent to (U) as systems.