Week 8
Taylor Series
Indefinite Integrals

$f(x)$	a	**Series**
$\sin(5x)$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k 5^{2k+1}}{(2k+1)!} x^{2k+1}$
$x^3 \cos x$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+3}$
$\sin(x-\pi)$	π	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x-\pi)^{2k+1}$
$\ln x$	1	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$
$\frac{1}{2-x}$	0	$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} x^k$
$\frac{1}{1+x}$	0	$\sum_{k=0}^{\infty} (-1)^k x^k$
$\frac{1}{1+x^2}$	0	$\sum_{k=0}^{\infty} (-1)^k x^{2k}$
$\arctan x$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$
$\frac{x+1}{x^2+x+1}$	0	$\sum_{k=0}^{\infty} \left(x^{3k} - x^{3k+2}\right)$
$\frac{1}{(1+x)^2}$	0	$\sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1}$
$\frac{1}{(1+x)(2-x)} = \frac{1}{3}\left(\frac{1}{1+x} + \frac{1}{2-x}\right)$	0	$\sum_{k=0}^{\infty} \frac{1}{3}\left((-1)^k + \frac{1}{2^{k+1}}\right) x^{2k}$

Theorem.

Generalized Binomial Theorem For $t, r \in \mathbb{R}$ such that $|t| < 1$, we have:

$$(1+t)^r = \sum_{k=0}^{\infty} \binom{r}{k} t^k$$
$$= 1 + rt + \frac{r(r-1)}{2!}t^2 + \frac{r(r-1)(r-2)}{3!}t^3 + \cdots,$$

where $\displaystyle\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$.

Example.

Find the Taylor series of $f(x) = \sqrt[3]{x-1}$ at 0.
Applying the Generalized Binonomial Theorem to $(1+t)^r$, where $t = -x$ and $r = 1/3$, we have:

$$f(x) = \sqrt[3]{x-1} = -(1-x)^{1/3}$$
$$= -\sum_{k=0}^{\infty} \binom{r}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{1/3}{k} (-1)^{k+1} x^k$$
$$= -1 + \frac{1}{3}x - \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2}x^2 + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3 - \cdots$$
$$= -1 + \frac{1}{3}x + \frac{1}{2!}\cdot\frac{2}{3^2}x^2 + \frac{1}{3!}\cdot\frac{2\cdot 5}{3^3}x^3 + \frac{1}{4!}\cdot\frac{2\cdot 5\cdot 8}{3^4}x^4 - \cdots$$

for $|x| < 1$. This is a power series centered at 0, hence it is the Taylor series of f at 0.

It is sometimes useful to use Taylor series to find limits which involve indeterminate forms.

Example.

- $$\lim_{x \to 0} \frac{\sin x - x - x^3}{x^3}$$
- $$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\arctan x}\right)$$

Indefinite Integrals

Definition.

If $F' = f$, we say that F is an **antiderivative** of f.

If two functions F and G are both antiderivatives of f over (a, b), then $F' = G' = f$, hence:

$$(F - G)' = F' - G' = 0.$$

By a corollary of the mean value theorem, this implies that $F - G$ is a constant function on (a, b). That is, there exists $C \in \mathbb{R}$, such that $(F - G)(x) = C$ for all $x \in (a, b)$.
Put differently, if F is an antiderivative of f over (a, b), then any antiderivative of f over (a, b) has the form $F + C$ for some constant function C.

Definition.

The collection of all antiderivatives of a function f is called the **indefinite integral** of f, denoted by:

$$\int f(x)\, dx.$$

We call $f(x)$ the **integrand** of $\int f(x)\, dx$.

If $F' = f$, we write:

$$\int f(x) dx = F + C,$$

where C denotes some arbitrary constant.
>

Example.

Since $\frac{d}{dx} x^2 = 2x$, we write:

$$\int 2x dx = x^2 + C.$$

Note that $x^2 + 17$ is also an antiderivative of $2x$, hence it is equally valid to write:

$$\int 2x dx = x^2 + 17 + C.$$

Some Properties of Indefinite Integrals

- $\int 0 \, dx = C$, where C is some constant.
- For $k \in \mathbb{R}$, we have $\int k \, dx = kx + C$. In particular,

$$\int dx = \int 1 \, dx = x + C.$$

- For $k \in \mathbb{R} \{-1\}$, we have:

$$\int x^k \, dx = \frac{x^{k+1}}{k+1} + C.$$

- $\int \frac{1}{x} \, dx = \ln|x| + C.$
  (This identity is not quite true. Will explain later.)
- $\int e^x \, dx = e^x + C.$
- $\int \cos x \, dx = \sin x + C.$
- $\int \sin x \, dx = -\cos x + C.$
- $\int \sec^2 x \, dx = \tan x + C.$
- $\int \sec x \tan x \, dx = \sec x + C.$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C.$
- For any functions f, g with antiderivatives F, G, respectively, we have:

$$\int (f(x) + g(x)) \, dx = F(x) + G(x) + C.$$

- For $k \in \mathbb{R}$, and any function f with antiderivative F, we have: $\int kf(x) \, dx = kF(x) + C.$

Observe that for any $a, b \in \mathbb{R}$, and differentiable function F, by the chain rule we have:

$$\frac{d}{dx} F(ax+b) = aF'(ax+b)$$

Hence, in general we have:

$$\int f(ax+b) \, dx = \frac{1}{a} F(ax+b) + C,$$

where F is an antiderivative of f, and C is some constant.

>

Example.

$$\int \sin(5x + \pi/4)\, dx = \frac{1}{5}\cos(5x + \pi/4) + C.$$

>

Example.

$$\int \left(x^3 + \frac{4}{x^{1/3}} + (x+7)^9 + e^{2x+1}\right)\, dx$$
$$= \frac{1}{4}x^4 + 4\left(\frac{3}{2}\right)x^{2/3} + \frac{1}{10}(x+7)^{10} + \frac{1}{2}e^{2x+1} + C.$$

>

Example.

$$\int \sin^2(x)\, dx = \int \left(\frac{1-\cos(2x)}{2}\right)\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\, dx$$
$$= \int \frac{1}{2}\, dx - \frac{1}{2}\int \cos(2x) dx$$
$$= \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

Similarly, it may be shown that:

$$\int \cos^2(x)\, dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

Integration by Substitution

Theorem.

If $F' = f$, and g is a differentiable function, then:

$$\boxed{\int f(g(x))g'(x)\, dx = F(g(x)) + C.}$$

Proof.

>

This is just the Chain Rule in reverse, since:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

>

In Leibniz Notation, the theorem may be formulaed as follows:
Let $u = g(x)$, then $\frac{du}{dx} = g'(x)$, and:

$$\begin{aligned}\int f(g(x))g'(x)\,dx &= \int f(u)\frac{du}{dx}\,dx \\ &= \int f(u)\,du = F(u) + C = F(g(x)) + C.\end{aligned}$$

Example.

Evaluate:

- $\int x^2 e^{x^3+4}\,dx$
- $\int \frac{t}{\sqrt{t+2}}\,dt$
- $\int \tan x\,dx$
- $\int \frac{x^5 + x^3 + x}{x^2+1}\,dx$

Integration by Parts

Let u, v be differentiable functions. Recall the Product Rule:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Take the indefinite integral (with respect to x) of both sides of the above equation, we have:

$$\int \frac{d}{dx}(uv)dx = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx,$$

which implies that:

$$\int d(uv) = \int v\,du + \int u\,dv.$$

Hence,

$$\boxed{\int u\,dv = (uv) - \int v\,du}$$

Example.

Evaluate:

- $\displaystyle\int xe^{3x}\,dx$
- $\displaystyle\int x^2e^x\,dx$
- $\displaystyle\int x^5e^x\,dx$
- $\displaystyle\int x^5\sin x\,dx$
- $\displaystyle\int \ln x\,dx$
- $\displaystyle\int e^x\sin x\,dx$
