Week 2
Functions

Sandwich Theorem Continued

Exercise.

1. Using the inequality $\dfrac{1}{\sqrt{n^2+n}} \le \dfrac{1}{\sqrt{n^2+r}} \le \dfrac{1}{\sqrt{n^2+1}}$ for $r = 1, 2, 3, \cdots, n$, prove that

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$$

Functions

Definition.

A function:

$$f : A \longrightarrow B$$

is a rule of correspondence from one set A (called the **domain**) to another set B (called the **codomain**).
Under this rule of correspondence, each element $x \in A$ corresponds to $f(x) \in B$, called the **value** of f at x.

In the context of this course, A is usually some subset (intervals, union of intervals) of $\mathbb{R}$, while B is often presumed to be $\mathbb{R}$.

Sometimes, the domain of a function is not explicitly given, and a function is simply defined by an expression in terms of an independent variable.

Example.

$$f(x) = \sqrt{\frac{x+1}{x-2}}$$

In this case, the domain of f is assumed to be the **natural domain** (or **maximal domain**, **domain of definition**), namely the largest subset of $\mathbb{R}$ on which the expression defining f is well-defined.

Graphs of Functions

For $f : A \longrightarrow B$ where A, B are subsets of $\mathbb{R}$, it is often useful to consider the **graph** of f, namely the set of all points (x, y) in the xy-plane where $x \in A$ and $y = f(x)$.
By definition, any function f takes on a unique value $f(x)$ for each x in its domain, hence the graph of f necessarily passes the so-called "**vertical line test**", namely, any vertical line which one draws in the xy-plane intersects the graph of f **at most once**.

The graph of a circle, for example, is not the graph of any function, since there are vertical lines which intersect the graph twice.

Exercise.

Graph the functions $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{4}{x} - 1$ together, to identify values of x for which

$$\frac{x}{2} > \frac{4}{x} - 1.$$

Confirm your answer by solving the inequality algebraically.

Answer.

The inequality holds if and only if:

$$x \in (-4, 0) \cup (2, \infty)$$

Definition.

Given two functions:

$$f, g : A \longrightarrow \mathbb{R},$$

- Their **sum/difference** is:

$$f \pm g : A \longrightarrow \mathbb{R},$$
$$(f \pm g)(a) := f(a) \pm g(a), \qquad \text{for all } a \in A;$$

- Their **product** is:

$$fg : A \longrightarrow \mathbb{R},$$
$$fg(a) := f(a)g(a), \qquad \text{for all } a \in A;$$

- and the **quotient function** $\dfrac{f}{g}$ is:

$$\frac{f}{g} : A' \longrightarrow \mathbb{R},$$
$$\frac{f}{g}(a) := \frac{f(a)}{g(a)}, \qquad \text{for all } a \in A',$$

  where

$$A' = \{a \in A : g(a) \neq 0\}.$$

Given two functions:

$$f : A \longrightarrow B, \quad g : B \longrightarrow C,$$

the **composite function** $g \circ f$ is defined as follows:

$$g \circ f : A \longrightarrow C,$$
$$(g \circ f)(a) := g(f(a)), \qquad \text{for all } a \in A.$$

The **range** or **image** of a function $f : A \longrightarrow B$ is the set of all $b \in B$ such that $b = f(a)$ for some $a \in A$.

In mathematical notation, we have:

$$\text{Image}(f) = \text{Range}(f) := \{b \in B \,:\, b = f(a) \text{ for some } a \in A\}.$$

Note that the range of f is not necessarily equal to the codomain B.

Definition.

If $\mathrm{Range}(f) = B$, we say that f is **surjective** or **onto**.

Definition.

If $f(a) \neq f(a')$ for all $a, a' \in \mathrm{Domain}(f)$ such that $a \neq a'$, we say that f is **injective** or **one-to-one**.

If $f : A \longrightarrow B$ is injective, then there exists an **inverse function**:

$$f^{-1} : \mathrm{Range}(f) \longrightarrow A$$

such that $f^{-1} \circ f$ is the **identity function** on A, and $f \circ f^{-1}$ is the identity function on $\mathrm{Range}(f)$, that is:

$$f^{-1}(f(a)) = a, \qquad \text{for all } a \in A,$$
$$f(f^{-1}(b)) = b, \qquad \text{for all } b \in \mathrm{Range}(f).$$

Example.

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$
$$f(x) := x^2, \quad x \in \mathbb{R}.$$

is not injective, hence it has no inverse.

On the other hand,

$$f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R},$$
$$f(x) := x^2, \quad x \in \mathbb{R}_{\geq 0};$$

is injective. It's range is $\mathrm{Range}(f) = \mathbb{R}_{\geq 0}$. Its inverse is:

$$f^{-1} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$$
$$f^{-1}(y) = \sqrt{y}, \quad y \in \mathbb{R}_{\geq 0}.$$

Similarly,

$$g : \mathbb{R}_{\leq 0} \longrightarrow \mathbb{R},$$
$$g(x) := x^2, \quad x \in \mathbb{R}_{\leq 0};$$

is also injective, with $\mathrm{Range}(g) = \mathbb{R}_{\geq 0}$, and inverse:

$$g^{-1}:\mathbb{R}_{\geq 0}\longrightarrow\mathbb{R}_{\leq 0}$$

$$f^{-1}(y)=-\sqrt{y},\quad y\in\mathbb{R}_{\geq 0}.$$

Piecewise Defined Functions

Example.

- $$f(x)=\begin{cases}-x+1 & \text{if} \quad -2\leq x<0\\ 3x & \text{if} \quad 0\leq x\leq 5\end{cases}$$

- The **absolute value function**

$$|x|=\begin{cases}-x & \text{if} \quad x<0\\ x & \text{if} \quad x\geq 0\end{cases}$$

Exercise.

Let $f:\mathbf{R}\longrightarrow\mathbf{R}$ be the function defined by:

$$f(x)=-3x+4-|x+1|-|x-1|$$

for any $x\in\mathbf{R}$.

1. Express the `explicit formula' of the function $f$ as that of a piecewise defined function, with one `piece' for each of $(-\infty,-1)$, $[-1,1]$, $(1,+\infty)$.
2. Sketch the graph of the function f.
3. Is f an injective function on $\mathbf{R}$? Justify your answer.
4. What is the image of $\mathbf{R}$ under the function f?

Limits of Functions on the Real Line

Let $f:A\longrightarrow\mathbb{R}$ be a function, where $A\subseteq\mathbb{R}$. Let a be a point on the real line such that f is defined on a neighborhood of a (though not necessarily at a itself).

Definition.

We say that the **limit** of f at a is L if for all $\varepsilon>0$, there exists $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever x satisfies $0<|x-a|<\delta$.

If f has a limit L at a, we write:

$$\lim_{x\to a}f(x)=L.$$

Note that the limit may exist even if a does not lie in the domain of f.

Definition.

Let $f : A \longrightarrow \mathbb{R}$ be a function, where $A \subseteq \mathbb{R}$ is unbounded towards $+\infty$ and/or $-\infty$. We say that the **limit** of f at ∞ (resp. $-\infty$) is L if for all $\varepsilon > 0$, there exists a $c \in \mathbb{R}$ such that $|f(x) - L| < \varepsilon$ whenever $x > c$ (resp. $x < c$).

If f has a limit L at ∞ (resp $-\infty$), we write:

$$\lim_{x \to \infty} f(x) = L \quad \left(\text{resp. } \lim_{x \to -\infty} f(x) = L\right)$$

Some useful identities:

In the following idenities, the symbol a can be either a real number or $\pm\infty$.

- For any constant $c \in \mathbb{R}$, we have $\lim\limits_{x \to a} c = c$.
- $\lim\limits_{x \to a} x = a$.
- If $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} g(x) = M$, then:
  - $\lim_{x \to a} (f \pm g)(x) = L \pm M$.
  - $\lim_{x \to a} fg(x) = LM$.
  - $$\lim_{x \to a} \frac{f}{g}(x) = \frac{L}{M}$$

    provided that $M \neq 0$.
- If $\lim_{x \to a} f(x) = L$, then:

  $$\lim_{x \to a} (f(x))^n = L^n \quad \text{for all } n \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

  and

  $$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad \text{for all odd positive integers } n.$$
- If $\lim_{x \to a} f(x) = L > 0$, then $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ for all $n \in \mathbb{N}$.

Example.

Compute the following limits, if they exist:

- $\lim_{x \to 4} \frac{2 - \sqrt{x}}{16 - x^2}$

- $\lim_{x \to -1} \frac{x^2 - 1}{x^2 - 5x - 6}$

One-Sided Limits

- We write $\lim_{x \to a^+} f(x) = L$ if $f(x)$ approaches L as x approaches a *from the right*. We call this L the **right limit** of f at a.
- Similarly, we write $\lim_{x \to a^-} f(x) = L$ if $f(x)$ approaches L as x approaches a *from the left*. We call this L the **left limit** of f at a.

The limit $\lim_{x \to a} f(x)$ is sometimes called the **double-sided** limit of f at a. It exists if and only if both one-sided limits exist and are equal to each other. In which case, we have:

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Exercise.

Define

$$f(x) = \begin{cases} x - 1 & \text{if } 1 \leq x \leq 2, \\ 2x + 3 & \text{if } 3 \leq x \leq 4, \\ x^2 & \text{otherwise.} \end{cases}$$

Compute $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$. Then, find $\lim_{x \to 2} f(x)$, if it exists.