Week 13
Integration

Recall:

Theorem.

Let $S(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ be a power series which converges on an open interval of the form $(a-r, a+r)$, $r > 0$, then the function $S(x)$ is differentiable on $(a-r, a+r)$, with:

$$S'(x) = \sum_{k=0}^{\infty} k a_k (x-a)^{k-1}$$

for all $x \in (a-r, a+r)$.

The theorem just cited works "in reverse", namely:

Theorem.

Let $S(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ be a power series which converges on an open interval of the form $(a-r, a+r)$, $r > 0$. Then, the power series:

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-a)^{k+1} = a_0(x-a) + \frac{a_1}{2}(x-a)^2 + \frac{a_2}{3}(x-a)^3 + \cdots$$

also converges on $(a-r, a+r)$, and:

$$\int S(x)\, dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-a)^{k+1} + C$$

over $(a-r, a+r)$, where C is an arbitrary constant. For $b \in (a-r, a+r)$, we have:

$$\int_a^b S(x)\, dx = \sum_{k=0}^{\infty} \int_a^b a_k (x-a)^k\, dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (b-a)^{k+1}.$$

Example.

The function:

$$F(x) = \int_0^x e^{-t^2}\,dt$$

is a differentiable function, but it has been proved that it is not an elementary function.

To describe F more explicitly, one can first consider the Taylor series of $f(x) = e^{-x^2}$ about $x = 0$:

$$S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k},$$

which converges to $f(x)$ for all $x \in \mathbb{R}$.

Using the theorem just stated, we see that:

$$F(x) = \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k}\,dt = \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^k}{k!} t^{2k}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} x^{2k+1}.$$

Example.

Given that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for all $x \in (-1, 1)$, find the Taylor series of $f(x) = \ln(1 + x^2)$ about $x = 0$.

>

Notice that f is an antiderivative of $g(x) = \frac{2x}{1+x^2}$. Since:

$$\frac{2x}{1+x^2} = 2x \cdot \frac{1}{1+x^2} = \sum_{k=0}^{\infty} 2(-1)^k x^{2k+1},$$

for all $x \in (-1, 1)$. we have:

$$f(x) = \sum_{k=0}^{\infty} \frac{2(-1)^k}{2k+2} x^{2k+2} + C$$

for all $x \in (-1, 1)$, for some constant $C \in \mathbb{R}$. Substituting $x = 0$ into both sides of the above equation, we have:

$$C = f(0) = \ln(1 + 0^2) = 0.$$

Hence, the Taylor series of f about $x = 0$ is:

$$\sum_{k=0}^{\infty} \frac{2(-1)^k}{2k+2} x^{2k+2}.$$

Example.

For each of the following functions f, find $F(x) := \int_0^x f(t)\,dt$ for all $x \in \mathbb{R}$. Then find $F'(x)$.

- $$f(x) = \begin{cases} 1 - x^2, & x \leq 1; \\ x - 1, & x > 1. \end{cases}$$
- $$f(x) = \begin{cases} x^2, & x \leq 1; \\ x, & x > 1. \end{cases}$$

A few words on t-substitution

Evaluate:

$$\int \frac{1}{1 + 2\cos x}\,dx$$

>

Let:

$$t = \tan\frac{x}{2}.$$

Then,

>

$$x = 2\arctan t,$$

$$dx = \frac{2}{1+t^2}dt$$

Moreover,

>

by the double-angle formula for the sine function, we have:

$$\begin{aligned} \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\ &= 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2} \\ &= \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} \\ &= \frac{2t}{1+t^2} \end{aligned}$$

>

Similarly, by the double-angle formula for the cosine function, we have:

$$\begin{aligned}\cos x &= 1 - 2\sin^2\frac{x}{2}\\ &= 1 - 2\tan^2\frac{x}{2}\cos^2\frac{x}{2}\\ &= 1 - \frac{2\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}}\\ &= 1 - \frac{2t^2}{1+t^2}\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

$$\cos x = \frac{1-t^2}{1+t^2}.$$

>
We have:

$$\begin{aligned}\int \frac{1}{1+2\cos x}\,dx &= \int \frac{1}{1+2\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}\,dt\\ &= \int \frac{2}{3-t^2}\,dt\\ &= \frac{1}{\sqrt{3}}\int\left(\frac{1}{\sqrt{3}+t}+\frac{1}{\sqrt{3}-t}\right)\,dt\\ &= \frac{1}{\sqrt{3}}(\ln|\sqrt{3}+t| - \ln|\sqrt{3}-t|) + C\\ &= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right| + C,\end{aligned}$$

where C is an arbitrary constant.

Improper Integral (Extracurricular Topic)

For a continuous function f defined on $[a, b)$ (where b can be ∞), we let:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.$$

Similarly, for a continuous function f defined on $(a, b]$ (where a can be $-\infty$), we let:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx.$$

- $\displaystyle\int_0^1 \frac{1}{x}\,dx$
- $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
- $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

- $\int_1^{\infty} \frac{1}{x^2}\, dx$
- $\int_1^{\infty} \frac{1}{x^p}\, dx$
- $\int_1^{\infty} \frac{\ln x}{x^2}\, dx$
- $\int_0^{\infty} e^{-x^2}\, dx$

If a function f is continuous on $[a, b]$ except at a point $c \in (a, b)$, we define:

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Note that the integrals on the right may themselves be improper integrals. We say that $\int_a^b f(x)\, dx$ is convergent if both integrals on the right are convergent.

Example.
$\int_0^2 \frac{1}{(x-1)^{2/3}}\, dx$

For a continuous function f on (a, b), where a can be $-\infty$ and b can be ∞, we fix any point $c \in (a, b)$ and define:

$$\int_a^b f(x)\, dx = \lim_{s \to a^+} \int_s^c f(x)\, dx + \lim_{t \to b^-} \int_c^t f(x)\, dx$$

We say that the integral is convergent if both limits are convergent.

Note that if both limits converge for one choice of c, then they both converge for any other choice of c. Hence, this definition of convergence is independent of the choice of c.

Example.
$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$
