THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2022-2023 Term 1
Suggested Solutions of WeBWork Coursework 8

If you find any errors or typos, please email us at
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1. (1 point)

Suppose that $f(x)$ and $g(x)$ are given by the power series
$f(x) = 6 + 4x + 2x^2 + 4x^3 + \cdots$
and
$g(x) = 2 + 2x + 2x^2 + 3x^3 + \cdots$.

Find the first few terms of the series for
$h(x) = f(x) \cdot g(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$.

$c_0 =$ ________

$c_1 =$ ________

$c_2 =$ ________

$c_3 =$ ________

Solution:
$c_0 = 6 \times 2 = 12$
$c_1 = 6 \times 2 + 2 \times 4 = 20$
$c_2 = 6 \times 2 + 4 \times 2 + 2 \times 2 = 24$
$c_3 = 6 \times 3 + 4 \times 2 + 2 \times 2 + 4 \times 2 = 38$

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2. (1 point) Let $f(x) = \begin{cases} \dfrac{\cos(4x^4) - 1}{x^7}, & x \neq 0; \\ 0, & x = 0. \end{cases}$
Given that f is infinitely differentiable at $x = 0$, evaluate the 9^{th} derivative of f at $x = 0$.
$f^{(9)}(0) =$ ____________

Solution:
It is known that

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - ... = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - ...$$

Then we have

$$\begin{aligned} f(x) &= \frac{\cos(4x^4) - 1}{x^7} \\ &= \frac{(1 - \frac{1}{2}(4x^4)^2 + \frac{1}{24}(4x^4)^4 - ...) - 1}{x^7} \\ &= \frac{-8x^8 + \frac{32}{3}x^{16} - ...}{x^7} \\ &= -8x + \frac{32}{3}x^9 - ... \end{aligned}$$

Therefore, we have

$$\frac{1}{9!}f^{(9)}(0) = \frac{32}{3}$$
$$f^{(9)}(0) = 3870720$$

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3. (1 point)
Evaluate the following indefinite integral $\int (13\sin(x) - 3\cos(x))\,dx =$ __________ $+C$

Solution:

$$\int 13\sin(x) - 3\cos(x)\;dx = 13\int \sin(x)\,dx - 3\int \cos(x)\,dx$$
$$= -13\cos(x) - 3\sin(x) + C$$

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4. (1 point) Calculate the following antiderivatives:

(a) $\int \frac{6}{x}\,dx =$ ____________________ $+C$.

(b) $\int (-5\sin x + 4\cos x)\,dx =$ ____________________ $+C$.

(c) $\int 3e^x\,dx =$ ____________________ $+C$.

Solution:

(a) $\int \frac{6}{x}\,dx = 6\,\ln|x| + C$.

(b) $\int (-5\sin x + 4\cos x)\;dx = -5\int \sin x\,dx + 4\int \cos x\,dx = 5\cos x + 4\sin x + C$.

(c) $\int 3e^x\,dx = 3e^x + C$.

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5. (1 point)
Using an upper-case "C" for any arbitrary constants, find the general indefinite integral:

$$\int \left(4x^2 + 3 + \frac{9}{x^2+1}\right) dx$$

Integral = ____________________

Solution:

$$\int (4x^2 + 3 + \frac{9}{x^2+1})dx = \frac{4}{3}x^3 + 3x + 9\arctan(x) + C$$

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6. (1 point) Evaluate the indefinite integral.

$$\int \left(4\sin(t) - 9\cos(t) - \sec^2(t) + 6e^t + \frac{3}{\sqrt{1-t^2}} + \frac{5}{1+t^2}\right) dt =$$

______________________ $+C$.

Solution:

$$\int (4\sin(t) - 9\cos(t) - \sec^2(t) + 6e^t + \frac{3}{\sqrt{1-t^2}} + \frac{5}{1+t^2})dt$$
$$= -4\cos(t) - 9\sin(t) - \tan(t) + 6e^t + 3\arcsin(t) + 5\arctan(t) + C$$

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7. (1 point)
Given that $f''(x) = \cos(x)$, $f'(\pi/2) = 2$ and $f(\pi/2) = 2$ find:

$f'(x) =$ ______
$f(x) =$ ______

Solution:

We begin by finding $f'(x)$.
The general solution is:

$f'(x) = \int \cos(x)dx = \sin(x) + C$

We choose C so that the condition $f'(\pi/2) = 2$ is satisfied. Since $f'(\pi/2) = \sin(\pi/2)+C = 1 + C = 2$, we have $C = 1$ and thus:

$$f'(x) = \sin(x) + 1.$$

Now, to find $f(x)$, we solve the integral $\int(\sin(x) + 1)dx$ to find the following general solution:

$$f(x) = \int \sin(x) + 1dx = -\cos(x) + x + D$$

We choose D so that the initial solution $f(\pi/2) = 2$ is satisfied. Since

$$f(\pi/2) = -\cos(\pi/2) + \frac{\pi}{2} + D = 0 + \frac{\pi}{2} + D = 2,$$

we have $D = 2 - \frac{\pi}{2}$ and the particular solution is:

$$f(x) = -\cos(x) + x + 2 - \frac{\pi}{2}$$

8. (1 point) Consider the function $f(x)$ whose second derivative is $f''(x) = 2x + 10\sin(x)$. If $f(0) = 2$ and $f'(0) = 2$, what is $f(x)$?

Answer: ____________________

Solution:
To solve this formula we will use the standard rule for the antiderivatives of a polynomial and the fact that the antiderivative of sin is $-\cos$ while the antiderivative of cos is sin. Furthermore, we will use the information given to us about $f'(0)$ and $f(0)$ to eliminate any arbitrary constants.

Finding the general antiderivative of $f''(x)$ gives a formula for $f'(x)$ as shown below:

$$f'(x) = x^2 - 10\cos(x) + C_1$$

We are told that $f'(0) = 2$ so that we can solve for C_1 in the expression above as shown.

$$f'(0) = (0)^2 - 10\cos(0) + C_1$$

$$2 = -10 + C_1$$

$$C_1 = 12$$

Therefore, the formula for $f'(x)$ is:

$$f'(x) = x^2 - 10\cos(x) + 12$$

Now, we take the antiderivative of $f'(x)$ to find the general formula for the function $f(x)$.

$$f(x) = \frac{1}{3}x^3 - 10\sin(x) + 12x + C_2$$

We are told that $f(0) = 2$ so that we can solve for C_2 as we did for C_1 above.

$$f(0) = \frac{1}{3}(0)^3 - 10\sin(0) + 12(0) + C_2$$

$$2 = C_2$$

Therefore, the specific value for the function $f(x)$ is given by:

$$f(x) = \frac{1}{3}x^3 - 10\sin x + 12x + 2$$

9. (1 point) Find the following indefinite integrals.

$$\int \frac{x}{\sqrt{x+7}}\, dx = \underline{\hspace{4cm}} + C$$

$$\int \frac{\cos(t)}{(7\sin(t)+8)^2}\, dt = \underline{\hspace{4cm}} + C$$

Solution:

$$\begin{aligned}\int \frac{x}{\sqrt{x+7}} dx &= \int \frac{x+7-7}{\sqrt{x+7}} dx \\ &= \int \frac{x+7}{\sqrt{x+7}} dx - \int \frac{7}{\sqrt{x+7}} dx \\ &= \int (x+7)^{\frac{1}{2}} d(x+7) - 2\times 7\times \sqrt{x+7} + C \\ &= \frac{2}{3}(x+7)^{\frac{3}{2}} - 2\times 7\times \sqrt{x+7} + C \\ &= \frac{2}{3}(x+7)\sqrt{x+7} - 14\sqrt{x+7} + C\end{aligned}$$

$$\begin{aligned}\int \frac{\cos(t)}{(7\sin(t)+8)^2} dt &= \int \frac{1}{(7\sin(t)+8)^2} d\sin(t) \\ &= -\frac{1}{7(7\sin(t)+8)} + C\end{aligned}$$

10. (1 point) Evaluate the integral using an appropriate substitution.

$$\int \frac{\sec^2(6\sqrt{x})}{\sqrt{x}}\, dx = \underline{\hspace{4cm}} + C$$

Solution:

For $u = 6\sqrt{x}$ we have $\frac{1}{3}\ du = \frac{dx}{\sqrt{x}}$ and hence;

$$\int \frac{\sec^2(6\sqrt{x})}{\sqrt{x}}\ dx = \frac{1}{3}\int \sec^2(u)\ \ du = \frac{1}{3}\ \tan(u) + C = \frac{1}{3}\tan\left(6\sqrt{x}\right) + C$$

11. (1 point)

Evaluate the indefinite integral

$$\int \frac{-6\sin(x)}{1+\cos^2(x)}\, dx$$

Note: Any arbitrary constants used must be an upper-case "C".

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Solution:

$$\int \frac{-6\sin(x)}{1+\cos^2(x)}dx = \int \frac{6}{1+\cos^2(x)}d\cos(x)$$
$$= 6\arctan(\cos(x)) + C$$

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12. (1 point)

Evaluate the indefinite integral.

$\int e^{4x}\sin(5x)dx =$ ______________________ $+C$.

Solution:

$$\begin{aligned}
&\int e^{4x}\sin(5x)\,dx \\
&= \int -\frac{1}{5}e^{4x}\,d\cos(5x) \\
&= -\frac{1}{5}e^{4x}\cos(5x) + \int \frac{1}{5}\cos(5x)\,de^{4x} \\
&= -\frac{1}{5}e^{4x}\cos(5x) + \int \frac{4}{25}e^{4x}\,d\sin(5x) \\
&= -\frac{1}{5}e^{4x}\cos(5x) + \frac{4}{25}e^{4x}\sin(5x) - \int \frac{4}{25}\sin(5x)\,de^{4x} \\
&= -\frac{1}{5}e^{4x}\cos(5x) + \frac{4}{25}e^{4x}\sin(5x) - \frac{16}{25}\int e^{4x}\sin(5x)\,dx
\end{aligned}$$

Then

$$\int e^{4x}\sin(5x)\,dx + \frac{16}{25}\int e^{4x}\sin(5x)\,dx = \frac{4}{25}e^{4x}\sin(5x) - \frac{1}{5}e^{4x}\cos(5x)$$
$$\int e^{4x}\sin(5x)\,dx = \frac{4}{41}e^{4x}\sin(5x) - \frac{5}{41}e^{4x}\cos(5x) + C$$

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13. (1 point)

Evaluate the integral

$$\int 1\ln(x^2-1)\,dx$$

Note: Use an upper-case "C" for the constant of integration.

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Solution:

$$\begin{aligned}
\int 1 \ln(x^2-1)dx &= \int \ln(x^2-1)dx \\
&= x\ln(x^2-1) - \int x d\ln(x^2-1) \\
&= x\ln(x^2-1) - \int x\frac{2x}{x^2-1}dx \\
&= x\ln(x^2-1) - 2\int \frac{x^2}{x^2-1}dx \\
&= x\ln(x^2-1) - 2\int \frac{x^2-1+1}{x^2-1}dx \\
&= x\ln(x^2-1) - 2\int 1+\frac{1}{x^2-1}dx \\
&= x\ln(x^2-1) - 2x - 2\int \frac{1}{x^2-1}dx \\
&= x\ln(x^2-1) - 2x - 2\int \frac{1}{(x-1)(x+1)}dx \\
&= x\ln(x^2-1) - 2x - 2\int \frac{1}{2}(\frac{1}{(x-1)} - \frac{1}{(x+1)})dx \\
&= x\ln(x^2-1) - 2x - \int \frac{1}{(x-1)} - \frac{1}{(x+1)}dx \\
&= x\ln(x^2-1) - 2x - \ln|x-1| + \ln|x+1| + C
\end{aligned}$$

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14. (1 point) **Note:** You can get full credit for this problem by just entering the final answer (to the last question) correctly. The initial questions are meant as hints towards the final answer and also allow you the opportunity to get partial credit.

Consider the indefinite integral $\displaystyle\int \frac{1}{\sqrt{1+(4x-6)^2}}\,dx$

Then the most appropriate substitution to simplify this integral is $x = g(t)$ where
$g(t) =$ ____________________
Note: We are using t as variable for angles instead of θ, since there is no standard way to type θ on a computer keyboard.

After making this substitution and simplifying (using trig identities), we obtain the integral $\displaystyle\int f(t)\,dt$ where
$f(t) =$ ____________________

This integrates to the following function of t
$\displaystyle\int f(t)\,dt =$ ____________________ $+C$

After substituting back for t in terms of x we obtain the following final form of the answer:
______________________________ $+C$

Solution:

Let $4x - 6 = \tan(t)$, then

$$g(t) = x = \frac{\tan(t) + 6}{4}$$

Since $dx = \frac{1}{4}\sec^2(t)dt$, then

$$\begin{aligned}
\int \frac{1}{\sqrt{1 + (4x-6)^2}} dx &= \int \frac{1}{4} \frac{1}{\sqrt{1 + \tan^2(t)}} \sec^2(t) dt \\
&= \int \frac{1}{4} \frac{1}{\sec(t)} \sec^2(t) dt \\
&= \int \frac{1}{4} \sec(t) dt
\end{aligned}$$

Thus we have

$$f(t) = \frac{1}{4}\sec(t)$$

This integrates to the following function of t:

$$\begin{aligned}
\int f(t)dt &= \int \frac{1}{4}\sec(t)dt \\
&= \frac{1}{4}\int \frac{\sec(t)(\sec(t) + \tan(t))}{\sec(t) + \tan(t)} dt \\
&= \frac{1}{4}\int \frac{\sec^2(t) + \sec(t)\tan(t)}{\sec(t) + \tan(t)} dt
\end{aligned}$$

Let $u = \sec(t) + \tan(t)$, $du = (\sec^2(t) + \sec(t)\tan(t))dt$. Then

$$\begin{aligned}
\int f(t)dt &= \frac{1}{4}\int \frac{1}{u} du \\
&= \frac{1}{4}\ln|u| \\
&= \frac{1}{4}\ln|\sec(t) + \tan(t)| + C
\end{aligned}$$

After substituting back for t in terms of x we obtain the following final form of the answer:

$$\int \frac{1}{\sqrt{1 + (4x-6)^2}} dx = \frac{1}{4}\ln|(4x-6) + \sqrt{1 + (4x-6)^2}| + C$$

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15. (1 point)

Evaluate the integral

$$\int \frac{8t^5}{\sqrt{t^2 + 2}} dt$$

Note: Use an upper-case "C" for the constant of integration.

Solution:

Sub $u = t^2 + 2$, then $du = 2t\,dt$

$$\begin{aligned}
&\int \frac{8t^5}{\sqrt{t^2+2}}\,dt \\
&= \int \frac{4t^4}{\sqrt{u}}\,du \\
&= \int \frac{4(u-2)^2}{\sqrt{u}}\,du \\
&= \int 4u^{3/2} - 16u^{1/2} + 16u^{-1/2}\,du \\
&= 4 \cdot \frac{2}{5}u^{5/2} - 16 \cdot \frac{2}{3}u^{3/2} + 16 \cdot 2u^{1/2} + C \\
&= \frac{8}{5}(t^2+2)^{5/2} - \frac{32}{3}(t^2+2)^{3/2} + 32(t^2+2)^{1/2} + C
\end{aligned}$$

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16. (1 point)
Evaluate the integral

$$\int 1\sqrt{3-2x-x^2}\,dx$$

Note: Use an upper-case "C" for the constant of integration.

Solution:

$$\begin{aligned}
&\int 1\sqrt{3-2x-x^2}\,dx \\
&= \int \sqrt{4-(x+1)^2}\,dx
\end{aligned}$$

Sub $x+1 = 2\sin(\theta)$. Then $dx = 2\cos(\theta)d\theta$

$$\begin{aligned}
&\int \sqrt{4-(x+1)^2}\,dx \\
&= \int \sqrt{4-4\sin^2(\theta)} \cdot 2\cos(\theta)\,d\theta \\
&= \int 4\cos^2(\theta)\,d\theta \\
&= \int 2 + 2\cos(2\theta)\,d\theta \\
&= 2\theta + \sin(2\theta) + C
\end{aligned}$$

Note that $\theta = \arcsin\left(\frac{x+1}{2}\right)$, and $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{x+1}{2}\right)\left(\sqrt{1-\left(\frac{x+1}{2}\right)^2}\right)$

Then

$$\begin{aligned} & 2\theta + \sin(2\theta) + C \\ &= 2\arcsin\left(\frac{x+1}{2}\right) + 2\left(\frac{x+1}{2}\right)\left(\sqrt{1-\left(\frac{x+1}{2}\right)^2}\right) + C \\ &= 2\arcsin\left(\frac{x+1}{2}\right) + \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + C \end{aligned}$$

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17. (1 point) The form of the partial fraction decomposition of a rational function is given below.

$$\frac{-(4x^2+3x+19)}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$A =$ _____ $B =$ _____ $C =$ _____

Now evaluate the indefinite integral.

$\int \frac{-(4x^2+3x+19)}{(x+1)(x^2+4)} dx =$ ____________________ $+D$, where D is an arbitrary constant.

Solution:

Multiplying by the least common denominator gives

$$-4x^2 - 3x - 19 = A(x^2+4) + (Bx+C)(x+1)$$

Rearranging terms on the right hand side, yields

$$-4x^2 - 3x - 19 = (A+B)x^2 + (B+C)x + 4A + C$$

Now we equate the coefficients:

$$\begin{aligned} A + B &= -4 \\ B + C &= -3 \\ 4A + C &= -19 \end{aligned}$$

Solving the system gives $A = -4,\ B = 0$ and $C = -3$ so the partial fraction decomposition is

$$\frac{-(4x^2+3x+19)}{(x+1)(x^2+4)} = -\frac{4}{x+1} - \frac{3}{x^2+4}$$

The definite integral is then

$$\int \frac{-(4x^2+3x+19)}{(x+1)(x^2+4)} dx = -4\ln\left(|x+1|\right) - \frac{3}{2}\arctan\frac{x}{2} + D$$

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18. (1 point) Evaluate the indefinite integral.

$\int \frac{x^3+4}{x^2+5x+6}\,dx =$ ____________________ $+C$.

Solution:

$$\int \frac{x^3+4}{x^2+5x+6}\,dx$$
$$= \int x-5+\frac{19x+34}{(x+2)(x+3)}\,dx$$

Suppose $\frac{19x+34}{(x+2)(x+3)} = \frac{A}{x+2}+\frac{B}{x+3}$, where A, B are constants. Then

$$\frac{19x+34}{(x+2)(x+3)} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$
$$= \frac{(A+B)x+(3A+2B)}{(x+2)(x+3)}$$

Then $A=-4, B=23$

$$\int x-5+\frac{19x+34}{(x+2)(x+3)}\,dx$$
$$= \int x-5+\frac{-4}{x+2}+\frac{23}{x+3}\,dx$$
$$= \frac{x^2}{2}-5x-4\ln|x+2|+23\ln|x+3|+C$$

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19. (1 point)

Evaluate the integral

$$\int \frac{5}{(x+a)(x+b)}\,dx$$

for the cases where $a=b$ and where $a \neq b$.

Note: For the case where $a=b$, use only a in your answer. Also, use an upper-case "C" for the constant of integration.

If $a=b$: ____________________

If $a \neq b$: ____________________

Solution:

When $a=b$,

$$\int \frac{5}{(x+a)^2}\,dx$$
$$= -\frac{5}{x+a}+C$$

When $a \neq b$,
Suppose $\dfrac{5}{(x+a)(x+b)} = \dfrac{A}{x+a} + \dfrac{B}{x+b}$, where A, B are constants.
Then

$$\begin{aligned}\frac{5}{(x+a)(x+b)} &= \frac{A(x+b)+B(x+a)}{(x+a)(x+b)} \\ &= \frac{(A+B)x+(Ab+Ba)}{(x+a)(x+b)}\end{aligned}$$

Then $A = \dfrac{-5}{a-b}, B = \dfrac{5}{a-b}$

$$\begin{aligned}&\int \frac{5}{(x+a)(x+b)}\,dx \\ &= \int -\frac{5}{(a-b)(x+a)} + \frac{5}{(a-b)(x+b)}\,dx \\ &= \frac{5}{b-a}(\ln|x+a| - \ln|x+b|) + C \\ &= \frac{5}{b-a}\ln|\frac{x+a}{x+b}| + C\end{aligned}$$

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20. (1 point)

Evaluate the integral

$$\int \frac{2\sqrt{t}}{1+\sqrt[3]{t}}\,dt$$

Note: Use an upper-case "C" for the constant of integration.

Solution: Sub $u^2 = \sqrt[3]{t}$. Then $\sqrt{t} = u^3$, $u = t^{\frac{1}{6}}$ and $dt = 6u^5 du$.

$$\begin{aligned}&\int \frac{2\sqrt{t}}{1+\sqrt[3]{t}}\,dt \\ &= 2\int \frac{u^3}{1+u^2}\cdot 6u^5\,du \\ &= 12\int \frac{u^8}{1+u^2}\,du \\ &= 12\int \frac{u^8+u^6-u^6+u^4-u^4+u^2-u^2+1-1}{1+u^2}\,du \\ &= 12\int \frac{u^6(1+u^2)-u^4(1+u^2)+u^2(1+u^2)-(u^2+1)+1}{1+u^2}\,du \\ &= 12\int [(u^6-u^4+u^2-1)+\frac{1}{1+u^2}]\,du \\ &= 12\cdot(\frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \arctan(u)) + C \\ &= \frac{12}{7}t^{\frac{7}{6}} - \frac{12}{5}t^{\frac{5}{6}} + 4t^{\frac{1}{2}} - 12t^{\frac{1}{6}} + 12\arctan(t^{\frac{1}{6}}) + C\end{aligned}$$

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21. (1 point)

Evaluate the integral

$$\int \frac{10}{x^2\sqrt{4x+1}}\,dx$$

Note: Use an upper-case "C" for the constant of integration.

Solution: Sub $u = \sqrt{4x+1}$. Then $x^2 = (\frac{u^2-1}{4})^2$, $dx = \frac{1}{2}udu$.

$$\begin{aligned}
&\int \frac{10}{x^2\sqrt{4x+1}}\,dx\\
&= 10\int \frac{1}{(\frac{u^2-1}{4})^2 \cdot u} \cdot \frac{1}{2}u\,du\\
&= 10\int \frac{8}{(u^2-1)^2}\,du = 10\int \frac{8}{(u+1)^2(u-1)^2}\,du
\end{aligned}$$

Suppose $\frac{8}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2}$, where A, B, C, D are constants.

Then $A = 2$, $B = 2$, $C = -2$, $D = 2$.

$$\begin{aligned}
&10\int \frac{8}{(u+1)^2(u-1)^2}\,du\\
&= 10\int (\frac{2}{u+1} + \frac{2}{(u+1)^2} - \frac{2}{u-1} + \frac{2}{(u-1)^2})\,du\\
&= 10 \cdot [2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1}] + C\\
&= 20 \cdot [\ln|\sqrt{4x+1}+1| - \frac{1}{\sqrt{4x+1}+1} - \ln|\sqrt{4x+1}-1| - \frac{1}{\sqrt{4x+1}-1}] + C.
\end{aligned}$$