THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2022-2023 Term 1
Suggested Solutions of WeBWork Coursework 5

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(1) Find the derivative of the function.

$$y = \sqrt{x}e^{(x^2)}(x^2+10)^{10}$$

$y' =$ _______________________

Solution:

$$\begin{aligned} y' &= (\sqrt{x})'(e^{x^2}(x^2+10)^{10}) + \sqrt{x}(e^{x^2}(x^2+10)^{10})' \\ &= (\sqrt{x})'(e^{x^2}(x^2+10)^{10}) + \sqrt{x}(e^{x^2})'(x^2+10)^{10} + \sqrt{x}(e^{x^2})\left[(x^2+10)^{10}\right]' \\ &= \frac{e^{x^2}(x^2+10)^{10}}{2\sqrt{x}} + \sqrt{x}e^{x^2}(2x)(x^2+10)^{10} + \sqrt{x}e^{x^2}10(x^2+10)^9(2x). \\ &= e^{x^2}\sqrt{x}\left[\frac{(x^2+10)^{10}}{2x} + 2x(x^2+10)^{10} + 20x(x^2+10)^9\right]. \end{aligned}$$

(2) Calculate the derivative of the following function.

$$f(x) = \frac{e^x}{(e^x+3)(x+2)}$$

$f'(x) =$ _______________________ .

Solution:
To compute $f'(x)$ we begin with quotient rule

$$f'(x) = \frac{(e^x+3)(x+2)\frac{d}{dx}[e^x] - e^x\frac{d}{dx}[(e^x+3)(x+2)]}{((e^x+3)(x+2))^2}.$$

Next, recall that $\frac{d}{dx}[e^x] = e^x$, and use the product rule to compute

$$\frac{d}{dx}[(e^x+3)(x+2)] = \frac{d}{dx}[e^x+3](x+2) + (e^x+3)\frac{d}{dx}[x+2]$$

which is

$$(e^x)(x+2) + (e^x+3)(1).$$

Therefore

$$f'(x) = \frac{(e^x+3)(x+2)\cdot e^x - e^x\cdot(e^x(x+2)+(e^x+3))}{((e^x+3)(x+2))^2}$$

and after factoring out e^x in the numerator, expanding $(e^x+3)(x+2) = xe^x + 2\cdot e^x + 3x + 3\cdot 2$, and distributing the minus sign, we get

$$f'(x) = \frac{e^x(xe^x + 3x + 2e^x + 6 - xe^x - 2e^x - e^x - 3)}{((e^x+3)(x+2))^2}$$

which simplifies to

$$f'(x) = \frac{e^x(3x - e^x + 3)}{((e^x + 3)(x + 2))^2}.$$

(3) Find the derivative of $f(y) = e^{e^{(y^4)}}$,

$f'(y) =$ ____________________________

Solution:

$$\begin{aligned} f'(y) &= \frac{d(e^{e^{y^4}})}{dy} \\ &= \frac{d(e^{e^{y^4}})}{d(e^{y^4})} \cdot \frac{d(e^{y^4})}{dy} \\ &= e^{e^{y^4}} \cdot \frac{d(e^{y^4})}{dy} \\ &= e^{e^{y^4}} \cdot \frac{d(e^{y^4})}{dy^4} \cdot \frac{d(y^4)}{dy} \\ &= e^{e^{y^4}} \cdot e^{y^4} \cdot 4y^3 \\ &= 4y^3 e^{y^4} e^{e^{y^4}} \end{aligned}$$

(4) Differentiate $g(x) = \ln\left(\frac{6-x}{6+x}\right)$.

Solution:

$$\begin{aligned} g'(x) &= \frac{6+x}{6-x} \cdot \left(\frac{6-x}{6+x}\right)' \\ &= \frac{6+x}{6-x} \cdot \frac{(-1)(6+x) - (6-x) \cdot 1}{(6+x)^2} \\ &= \frac{-6 - x - 6 + x}{(6+x)(6-x)} \\ &= \frac{12}{x^2 - 36}. \end{aligned}$$

(5) Find $\frac{dr}{dx}$ if

$$r = \frac{\ln(9x)}{x^2 \ln(x^2)} + \left(\ln\left(\frac{4}{x}\right)\right)^3$$

$\frac{dr}{dx}$= ____________________________________

Solution: By the chain rule,

$$(\ln(9x))' = \frac{1}{9x} \cdot (9x)' = \frac{1}{9x} \cdot 9 = \frac{1}{x},$$

and by the product rule and chain rule,

$$(x^2\ln(x^2))' = (2x)\ln(x^2) + x^2 \cdot \frac{1}{x^2} \cdot (x^2)' = (2x)\ln(x^2) + x^2 \cdot \frac{1}{x^2} \cdot (2x) = (2x)\ln(x^2) + 2x$$

Applying the quotient rule to the first summand involves an application of the product rule.

$$\begin{aligned}
\left(\frac{\ln(9x)}{x^2\ln(x^2)}\right)' &= \frac{(\ln(9x))' \cdot (x^2\ln(x^2)) - (\ln(9x)) \cdot (x^2\ln(x^2))'}{(x^2\ln(x^2))^2} \\
&= \frac{(\frac{1}{x}) \cdot (x^2\ln(x^2)) - (\ln(9x)) \cdot ((2x)\ln(x^2) + 2x)}{(x^2\ln(x^2))^2} \\
&= \frac{x\ln(x^2) - \ln(9x)(2x)\ln(x^2) - \ln(9x)(2x)}{(x^2\ln(x^2))^2} \\
&= \frac{1}{x^3\ln(x^2)} - \frac{2\ln(9x)}{x^3\ln(x^2)} - \frac{2\ln(9x)}{x^3\ln^2(x^2)}
\end{aligned}$$

Then apply the power and chain rules to the second summand.

$$\begin{aligned}
((\ln\left(\frac{4}{x}\right))^3)' &= 3(\ln\left(\frac{4}{x}\right))^2 \cdot \frac{x}{4} \cdot \frac{-4}{x^2} \\
&= -\frac{3\ln^2\left(\frac{4}{x}\right)}{x}
\end{aligned}$$

By the sum rule, your answer should be equivalent to the expression

$$\frac{1}{x^3\ln(x^2)} - \frac{2\ln(9x)}{x^3\ln(x^2)} - \frac{2\ln(9x)}{x^3\ln^2(x^2)} - \frac{3\ln^2\left(\frac{4}{x}\right)}{x}.$$

(6) Find $f'(x)$ and $f'(0)$ where:

$$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Find the derivative of $f(x)$ for x not equal 0.

$f'(x) =$ __________

(b) If the derivative does not exist enter DNE.

$f'(0) =$ __________

Solution:

(a) Applying the product rule to $x^2\sin(\frac{1}{x})$ gives

$$f'(x) = 2x\sin(\frac{1}{x}) + x^2(\cos(\frac{1}{x}) \cdot (\frac{-1}{x^2}))$$

,
that is,

$$f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

.

(b) Using the definition of the derivative we find that:

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} h^2 \sin\left(\frac{1}{h}\right)\frac{1}{h} \\ &= \lim_{h\to 0} h\sin(\frac{1}{h}) \\ &= 0. \end{aligned}$$

(The last step above is due to the squeeze theorem).

(7) Let $f(x) = |x|\ln(2-x)$. Find $f'(x)$.

$$f'(x) = \begin{cases} ? & \text{if} \quad x < c \\ ? & \text{if} \quad x = c \\ ? & \text{if} \quad c < x < d \end{cases}$$

Solution: One can find that $c = 0$, $d = 2$.

$x < 0$,

$$\begin{aligned} f(x) &= -x\ln(2-x), \\ f'(x) &= -\ln(2-x) + \frac{x}{2-x}. \end{aligned}$$

$0 < x < 2$,

$$\begin{aligned} f(x) &= x\ln(2-x), \\ f'(x) &= \ln(2-x) - \frac{x}{2-x}. \end{aligned}$$

$x = 0$,

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h\ln(2-h)-0}{h} = \lim_{h\to 0^+}\ln(2-h) = \ln 2.$$

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h\ln(2-h)-0}{h} = \lim_{h\to 0^-} -\ln(2-h) = -\ln 2.$$

Since the limit of $\frac{f(h)-f(0)}{h}$ as $h \to 0$ doesn't exist, the derivative doesn't exist at $x = 0$.

(8) (a) If $f(x) = |\sin x|$, find $f'(x)$.

(b) Where is $f(x)$ non-differentiable? Please give the smallest positive value of x.

(c) If $g(x) = \sin|x|$, find $g'(x)$.

(d) Where is $g(x)$ non-differentiable?

Solution:

Since the derivative of the absolute value function $h(x) = |x|$ is that

$$h'(x) = \frac{x}{|x|}, \qquad x \neq 0.$$

(a)By the chain rule,

$$f'(x) = \frac{\sin x}{|\sin x|} \cdot (\sin x)' = \frac{\sin x}{|\sin x|} \cdot \cos x.$$

(b) Since $|\sin x| \neq 0$, So, you can solve the smallest positive value of x is π.

(c)Similarly to (a), by the chain rule,

$$g'(x) = \cos|x| \cdot (|x|)' = \cos|x| \cdot \frac{x}{|x|}.$$

(d)From (c), you can know that $g(x)$ is non-differentiable at $x = 0$.

(9) Compute $f'(x)$, $f''(x)$, $f'''(x)$, and then state a formula for $f^{(n)}(x)$, when

$$f(x) = -\frac{4}{x}$$

$f'(x) =$ ______________
$f''(x) =$ ______________
$f'''(x) =$ ______________
$f^{(n)}(x) =$ ______________

Solution:

$$f'(x) = \frac{d}{dx}\left[-\frac{4}{x}\right] = \frac{(-1)(-4)}{x^2} = \frac{4}{x^2},$$

$$f''(x) = \frac{d}{dx}\left[\frac{4}{x^2}\right] = \frac{(-2)(-1)(-4)}{x^3} = -\frac{8}{x^3},$$

$$f'''(x) = \frac{d}{dx}\left[-\frac{8}{x^3}\right] = \frac{(-3)(-2)(-1)(-4)}{x^4} = \frac{24}{x^4},$$

Observing the pattern we get that,

$$f^{(n)}(x) = \frac{4(-1)^{n+1}(n!)}{x^{n+1}}$$

(10) Find $\frac{dy}{dx}$ if

$$4x^3y^2 - 2x^2y = 6.$$

Express your answer in terms of x, y if necessary.

$\frac{dy}{dx} =$ ______________

Solution: Taking the derivative with respect to x we get

$$0 = 12x^2y^2 + 8x^3y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx},$$

or

$$4xy - 12x^2y^2 = \left(8x^3y - 2x^2\right)\frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx} = \frac{4xy - 12x^2y^2}{8x^3y - 2x^2}.$$

(11) Find $\frac{dy}{dx}$, if $y = \ln(9x^2 + 7y^2)$.

$\frac{dy}{dx} =$ _________

Solution: Writing the given equation as $e^y = 9x^2+7y^2$, and then differentiating implicitly with respect to x, gives

$$e^y\frac{dy}{dx} = 18x + 14y\frac{dy}{dx},$$

or

$$(e^y - 14y)\frac{dy}{dx} = 18x.$$

Therefore,

$$\frac{dy}{dx} = \frac{18x}{e^y - 14y}.$$

Note: Were the equation not revised before differentiating, the answer

$$\frac{dy}{dx} = \frac{18x}{9x^2 + 7y^2 - 14y}$$

would result.

(12) Consider the following function: $y = x^{x^2}$.

$$\frac{dy}{dx} =$$

Solution: For $y = x^{x^2}$ one has $\ln y = x^2 \ln x$. (This method is **standard** by using logarithm to transfer a power into a product.) So, by the chain rule we get that

$$(\ln y)' = \frac{1}{y}\cdot y' = (x^2\ln x)' = 2x\ln x + x^2\cdot\frac{1}{x} = 2x\ln x + x,$$

which implies that

$$\frac{dy}{dx} = y' = y(2x\ln x + x) = x^{x^2}(2x\ln x + x) = x^{x^2+1}(2\ln x + 1).$$

(13) If $f(x) = \cos(\sin(x^2))$, then $f'(x) =$ _______________

Solution:

$$\begin{aligned} f'(x) &= -\sin(\sin(x^2))\cdot(\sin(x^2))' \\ &= -\sin(\sin(x^2))\cdot\cos(x^2)\cdot(x^2)' \\ &= -\sin(\sin(x^2))\cdot\cos(x^2)\cdot(2x) \end{aligned}$$

(14) Let $f(x) = \dfrac{1}{\left(x^3 - \sec(3x^2 - 8)\right)^3}$. Find $f'(x)$.

$f'(x) =$ ________________________

Solution: Taking $u(x) = x^3 - \sec(3x^2 - 8)$, we know that

$$\frac{du}{dx} = 3x^2 - 6x\sec\big(3x^2 - 8\big)\tan\big(3x^2 - 8\big).$$

So, by the chain rule,

$$\begin{aligned}\frac{d}{dx}\frac{1}{\left(x^3 - \sec(3x^2 - 8)\right)^3} &= \frac{d}{dx}\frac{1}{u^3} \\ &= -\frac{3}{u^4}\frac{du}{dx} \\ &= -\frac{3}{\left(x^3 - \sec(3x^2 - 8)\right)^4} \cdot \Big(3x^2 - 6x\sec\big(3x^2 - 8\big)\tan\big(3x^2 - 8\big)\Big).\end{aligned}$$

(15) A parabola is defined by the equation

$$x^2 - 2xy + y^2 + 2x - 6y + 21 = 0$$

The parabola has horizontal tangent lines at the point(s) ________.

The parabola has vertical tangent lines at the point(s) ________.

Solution: Differentiating implicitly with respect to x gives

$$2x - 2y - 2x\,\frac{dy}{dx} + 2y\,\frac{dy}{dx} + 2 - 6\,\frac{dy}{dx} = 0,$$

or

$$(y - x - 3)\,\frac{dy}{dx} = y - x - 1,$$

and so

$$\frac{dy}{dx} = \frac{y - x - 1}{y - x - 3}.$$

The tangent line to the parabola is horizontal where $\frac{dy}{dx} = 0$, i.e., where $x - y = -1$. The equation of the parabola can be written in the form

$$(x - y)^2 + 2(x - y) + 21 - 4y = 0,$$

and $x - y = -1$ gives $20 = 4y$, or $y = 5$, and $x = 4$. Hence, the tangent line to the parabola is horizontal at the point $(4, 5)$ and nowhere else.

The tangent line to the the parabola is vertical where

$$0 = \frac{dx}{dy} = \frac{y - x - 3}{y - x - 1},$$

i.e., where $x - y = -3$. Together with the last displayed equation of the parabola, this gives $24 - 4y = 0$, or $y = 6$, and $x = 3$. Hence, the tangent line to the parabola is vertical at the point $(3, 6)$ and nowhere else.

(16) Let $x^3 + y^3 = 28$. Find $y''(x)$ at the point $(3, 1)$.

$y''(3) =$ ______________

Solution: Differentiting the equation implicitly with respect to x, we get

$$3x^2 + 3y^2 y' = 0$$

Solving for y' gives

$$y' = -\frac{x^2}{y^2}$$

To find y'' we differentiate this expression for y' using the quotient rule and remembering that y is a function of x:

$$y'' = -\frac{y^2 \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(y^2)}{(y^2)^2} = -\frac{y^2 \cdot 2x - x^2(2yy')}{y^4}$$

If we now substitute $y' = -\frac{x^2}{y^2}$ into this expression, we get

$$y'' = -\frac{2xy^2 - 2x^2 y\left(-\frac{x^2}{y^2}\right)}{y^4} = -\frac{2xy^3 + 2x^4}{y^5} = -\frac{2x(y^3 + x^3)}{y^5}$$

But the values of x and y must satisfy the original equation $x^3 + y^3 = 28$. So this expression simplifies to

$$y'' = -\frac{56x}{y^5}$$

Substituting $x = 3$ and $y = 1$ gives

$$y''(3) = -168$$

(17) Let $f(x) = \dfrac{4x^3}{(5-2x)^4}$.
Find the equation of the line tangent to the graph of f at $x = 2$.

Tangent line: $y =$ ______________________

Solution: Differentiating gives

$$\begin{aligned} f'(x) &= \frac{12x^2(5-2x)^4 - 4x^3 \cdot 4(5-2x)^3(-2)}{(5-2x)^8} \\ &= \frac{12x^2(5-2x) - 4x^3 \cdot 4 \cdot (-2)}{(5-2x)^5} \\ &= \frac{60x^2 + 8x^3}{(5-2x)^5} \end{aligned}$$

and hence the slope of the tangent line of the graph at $x = 2$ is $f'(2) = 304$. Since $f(2) = 32$ and the point $(2, 32)$ is also on this line, we know the tangent line $y - 32 = 304(x - 2)$, that is, $y = 304(x - 2) + 32$.

(18) Find all points on the graph of the function $f(x) = \sin 2x - 2\sin x, 0 \le x < \pi$ at which the tangent line is horizontal. List the x-values below, separating them with commas.

$$x = \_\_\_\_\_\_\_\_.$$

Solution: Differentiating with respect to x gives $f'(x) = 2\cos 2x - 2\cos x$. The tangent line to the graph of the function $f(x) = \sin 2x - 2\sin x, 0 \le x < \pi$ is horizontal where $f'(x) = 0$, this implies that

$2\cos 2x - 2\cos x = 0,\ 0 \le x < \pi,$

$\Longrightarrow \quad \cos 2x - \cos x = 0,\ 0 \le x < \pi,$

$\Longrightarrow \quad 2(\cos x)^2 - \cos x - 1 = 0,\ 0 \le x < \pi,$

$\Longrightarrow \quad \cos x = \frac{-1}{2} \quad \text{or} \quad 1,\ 0 \le x < \pi,$

$\Longrightarrow \quad x = \frac{2\pi}{3} \quad , \quad 0.$

Then $x = \frac{2\pi}{3} \quad , \quad 0.$

(19) If the equation of motion of a particle is given by $s(t) = A\cos(wt + d)$, the particle is said to undergo simple harmonic motion. Assume $0 \le d < \pi$

(a) Find the velocity of the particle at time t.

(b) What is the smallest positive value of t for which the velocity is 0 ? Assume that w and d are positive.

(a) $v(t) =$________ .

(b) $t =$________ .

Solution: (a) Differentiating respect to x gives: $v(t) = s'(t) = -Aw\sin(wt+d)$

(b) By (a), $v(t) = 0$ implies $\sin(wt + d) = 0$, then $wt + d = n\pi$, where n is integer. Since $0 \le d < \pi$, the smallest positive vale of t for which the velocity is 0 is

$$t = \frac{\pi - d}{w}.$$

(20) $\frac{d^4}{dx^4}\left(\frac{3x^4}{1-x}\right) = \_\_\_\_\_\_\_\_.$

Solution: You could use the quotient rule 4 times directly, but I will give another solution to you here. Since

$$3x^4 = (-3x^3 - 3x^2 - 3x - 3)(1 - x) + 3,$$

So

$$\frac{3x^4}{1-x} = -3x^3 - 3x^2 - 3x - 3 + \frac{3}{1-x},$$

By the power rule, we can know that

$$\frac{d^4}{dx^4}(-3x^3 - 3x^2 - 3x - 3) = 0,$$

So, by the sum rule,

$$\frac{d^4}{dx^4}(\frac{3x^4}{1-x}) = 0 + \frac{d^4}{dx^4}(\frac{3}{1-x}),$$

It is easy to know that

$$\frac{d}{dx}(\frac{3}{1-x}) = \frac{3}{(1-x)^2},$$

and

$$\frac{d^2}{dx^2}(\frac{3}{1-x}) = \frac{6}{(1-x)^3},$$

and

$$\frac{d^3}{dx^3}(\frac{3}{1-x}) = \frac{18}{(1-x)^4}.$$

Hence,

$$\frac{d^4}{dx^4}(\frac{3x^4}{1-x}) = \frac{d^4}{dx^4}(\frac{3}{1-x}) = \frac{72}{(1-x)^5}.$$

(21) Find a formula for $f^{(101)}(x)$ if $f(x) = \dfrac{1}{9x-1}$.

$f^{(101)}(x) =$ ______________

Solution: Differentiating respect to x gives:

$$f'(x) = \frac{-9}{(9x-1)^2}.$$

And

$$f''(x) = \frac{162}{(9x-1)^3}.$$

$$f'''(x) = \frac{-4374}{(9x-1)^4}.$$

Observing this pattern, we have

$$f^n(x) = (-1)^n 9^n \frac{n!}{(9x-1)^{n+1}}.$$

Now we let $n = 101$ then we get

$$f^{(101)}(x) = \frac{-101! \cdot 9^{101}}{(9x-1)^{102}}.$$