THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2021-2022 Term 1
Suggested Solutions of WeBWork Coursework 2

(1) The function $h(x)$ is continuous at every number in its domain. State domain.

$$h(x) = \frac{\sin(3x)}{x+3}$$

Solutions:
Since $x+3$ is on the denominator, $x+3 \neq 0$, which implies $x \neq -3$. For any $x \neq -3$, $\sin(3x)$ is continuous, $x+3$ is continuous and nonzero. Hence, the domain of $h(x)$ is $(-\infty, -3) \cup (-3, \infty)$.

(2) Find the domain of the function

$$f(x) = \frac{\sqrt{5+2x}}{x^2-100}$$

Solutions:
Since we take the square root of $5+2x$, $5+2x \geq 0$, which implies $x \geq -\frac{5}{2}$. Since x^2-100 is on the denominator, $x^2-100 \neq 0$, which implies $x \neq \pm 10$. Hence, the domain of $f(x)$ is $(-2.5, 10) \cup (10, \infty)$.

(3) The domain of the function

$$g(x) = \log_a (x^2 - 25)$$

is $(-\infty, \_)$ and $(\_, \infty)$
Solutions:
Since we take the logarithm of x^2-25, $x^2-25 > 0$, which implies $x < -5$ or $x > 5$. Hence, the domain of $g(x)$ is $(-\infty, -5) \cup (5, \infty)$.

(4) Given that $f(x) = \frac{1}{x}$ and $g(x) = 2x+4$, calculate $f \circ g(x)$, $g \circ f(x)$, $f \circ f(x)$, $g \circ g(x)$ and find their domains.
Solutions:
$f \circ g(x) = f(g(x)) = f(2x+4) = \frac{1}{2x+4}$.
The domain of $f \circ g(x)$ is $(-\infty, -2) \cup (-2, \infty)$.
$g \circ f(x) = g(f(x)) = g(\frac{1}{x}) = \frac{2}{x} + 4$.
The domain of $g \circ f(x)$ is $(-\infty, 0) \cup (0, \infty)$.
$f \circ f(x) = f(f(x)) = \frac{1}{\frac{1}{x}}$.
The domain of $f \circ f(x)$ is $(-\infty, 0) \cup (0, \infty)$.
(Remark: $f \circ f(x) = x$ on $(-\infty, 0) \cup (0, \infty)$, but it is not well defined at $x = 0$)
$g \circ g(x) = g(g(x)) = 2(2x+4) + 4 = 4x + 12$.
The domain of $g \circ g(x)$ is $(-\infty, \infty)$.

(5) Given the functions $f(x) = \frac{x-5}{x-3}$ and $g(x) = \sqrt{x+4}$, find the domains of f, g, $f+g$, $\frac{f}{g}$, $\frac{g}{f}$, $f \circ g$, $g \circ f$.

Solutions:

The domain of f is $(-\infty, 3) \cup (3, \infty)$.

The domain of g is $[-4, \infty)$.

$(f+g)(x) = \frac{x-5}{x-3} + \sqrt{x+4}$.

The domain of $f+g$ is $[-4, 3) \cup (3, \infty)$.

$\frac{f}{g}(x) = \frac{x-5}{(x-3)\sqrt{x+4}}$.

The domain of $\frac{f}{g}$ is $(-4, 3) \cup (3, \infty)$.

$\frac{g}{f}(x) = \frac{\sqrt{x+4}}{\frac{x-5}{x-3}}$.

The domain of $\frac{g}{f}$ is $[-4, 3) \cup (3, 5) \cup (5, \infty)$.

$f \circ g(x) = f(g(x)) = f(\sqrt{x+4}) = \frac{\sqrt{x+4}-5}{\sqrt{x+4}-3}$.

Since we take the square root of $x+4$, $x \geq -4$. Since $\sqrt{x+4} - 3$ is on the denominator, $\sqrt{x+4} - 3 \neq 0$,which implies $x \neq 5$.

Hence, the domain of $f \circ g$ is $[-4, 5) \cup (5, \infty)$.

$g \circ f(x) = g(f(x)) = \sqrt{\frac{x-5}{x-3} + 4}$.

Since $x - 3$ is on the denominator, $x \neq 3$. Since we take square root of $\frac{x-5}{x-3} + 4$, $\frac{x-5}{x-3} + 4 \geq 0$, which implies $x \geq 3.4$ or $x < 3$. Hence, the domain of $g \circ f$ is $(-\infty, 3) \cup [3.4, \infty)$.

(6) Use the graph to find the missing values

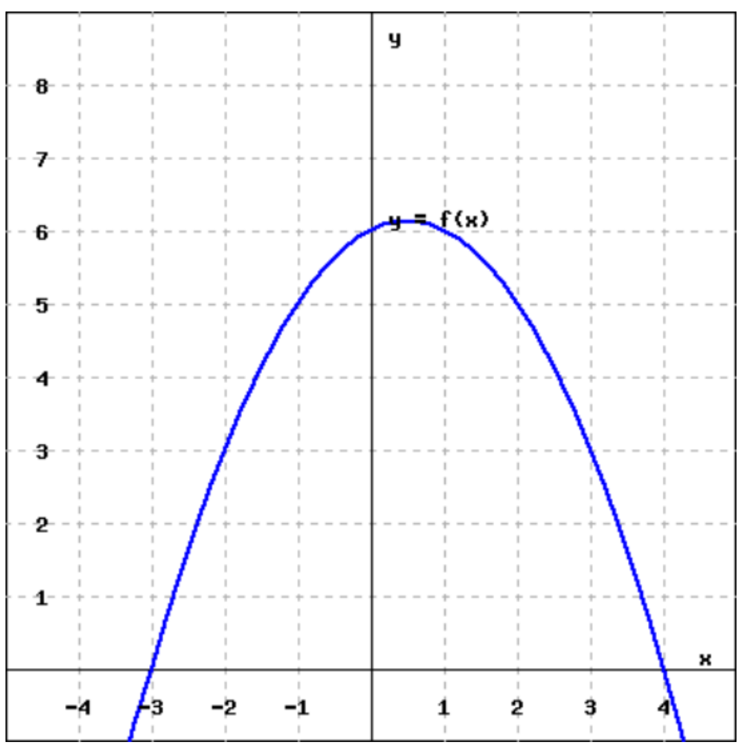

FIGURE (1) 6

Solutions:
$f(0) = 6$, $f(-3 \textit{ or } 4) = 0$.

(7) Suppose $f(x) = 4x - 9$ and $g(y) = \frac{y}{4} + \frac{9}{4}$.

(a) Find the composition $g(f(x))$.

(b) Find the composition $f(g(y))$.

(c) Are the functions f and g inverse to each other?

Solutions:

(a) $g(f(x)) = g(4x - 9) = \frac{4x-9}{4} + \frac{9}{4} = x$.

(b) $f(g(y)) = f(\frac{y}{4} + \frac{9}{4}) = 4 \times \frac{y}{4} + \frac{9}{4} - 9 = y$.

(c) Yes.

(8) Find the inverse function to $y = f(x) = 7x + 4$.
Solutions:
Expressing x in terms of y, we have $x = \frac{y}{7} - \frac{4}{7}$. Hence, $x = g(y) = \frac{y}{7} - \frac{4}{7}$.

(9) Find the inverse function to $y = f(x) = \frac{8-7x}{6-2x}$.
Solutions:
We express x in terms of y.

$$y(6 - 2x) = 8 - 7x$$

$$6y - 2xy = 8 - 7x$$

$$6y - 8 = x(2y - 7)$$

$$x = \frac{8 - 6y}{7 - 2y}$$

Hence, $x = g(y) = \frac{8-6y}{7-2y}$.

(10) Find the inverse function (if it exists) of $f(x) = ln(5 - 4x)$. if the function is not invertible, enter NONE.
Solutions:
Start with our property of inverse functions $f(f^{-1}(x)) = x$, and substitute y for

$f^{-1}(x)$ to get $f(y) = x$. Now, we can solve it like following:

$$\begin{aligned} x &= f(y) \\ &= ln(5-4y) \\ \Rightarrow e^x &= 5-4y \\ \Rightarrow y &= \frac{5-e^x}{4}. \end{aligned}$$

Thus $f^{-1} = \frac{5-e^x}{4}$ is the inverse function.

(11) Match the functions with their graphs. Enter the letter of the graph below which corresponds to the function.
1. $f(x) = 2$, if $x \leq -1$, $f(x) = x^2$, if $x > -1$.
2. $f(x) = -1$, if $x < 2$, $f(x) = 1$, if $x \geq 2$.
3. $f(x) = 1$, if$x \leq 1$, $f(x) = x + 1$, if$x > 1$.
4. $f(x) = x$, if$x \leq 0$, $f(x) = x + 1$, if$x > 0$.

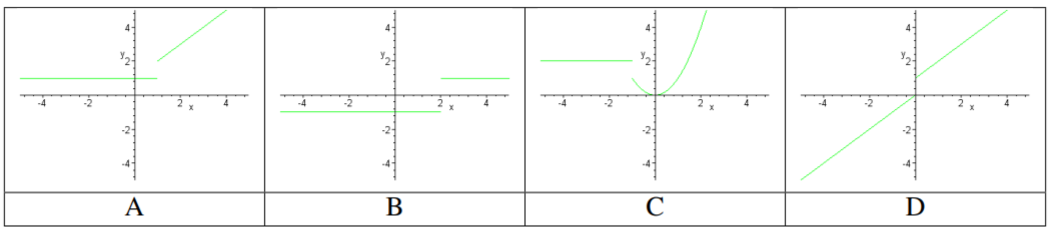

FIGURE (2) 11

Solutions:
1-C; 2-B; 3-A; 4-D.

(12) Part 1: Evaluate the limit

$$\lim_{x\to 4} \frac{x^2+x-20}{x-4} = \lim_{x\to 4} \_\_\_\_ = \_\_\_\_.$$

Part 2: Follow-up question
Solutions:
For $x \neq 4$ and $x \to 4$, we can transform the fraction into $\frac{(x-4)(x+5)}{x-4} = x - 5$. And therefore the limit is a finite number which is just the value $x - 5 = 4 - 5 = -1$.

(13) Evaluate the limit

$$\lim_{b\to 1} \frac{\frac{1}{b}-1}{b-1}.$$

Solutions:
Since $b \neq 0$, $\lim_{b\to 1} \frac{\frac{1}{b}-1}{b-1} = \lim_{b\to 1} \frac{1-b}{b^2-b}$.
For all $b \neq 1$, $b \to 1$, $\lim_{b\to 1} \frac{1-b}{b^2-b} = \lim_{b\to 1} \frac{-1}{b} = -1$.
Therefore, the limit is -1.

(14) Let a be a positive real number. Evaluate the limit:

$$\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{4(x-a)} = \_\_\_\_$$

Solutions:
For any $x \neq a, x \to a$,

$$\begin{aligned} \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{4(x-a)} &= \lim_{x \to \infty} \frac{\sqrt{x} - \sqrt{a}}{4(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} \\ &= \lim_{x \to a} \frac{1}{4(\sqrt{x} + \sqrt{a})} \\ &= \frac{1}{8\sqrt{a}}. \end{aligned}$$

(15) Determine whether the sequence $a_n = \frac{1^3}{n^4} + \frac{2^3}{n^4} + \cdots + \frac{n^3}{n^4}$ converges or diverges. If it converges, find the limit. Note,

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Converges (yes/no):
Limit:
Solutions:
yes, $\frac{1}{4}$.

$$a_n = \frac{\sum_{i=1}^{n} i^3}{n^4} = \frac{\frac{n^2(n+1)^2}{4}}{n^4} = \frac{(n+1)^2}{4n^2}$$
$$\lim_{n \to \infty} (1 + \frac{1}{n})^2 = 1.$$

Therefore the limit is $\frac{1}{4}$.

(16) Determine whether the sequences are increasing, decreasing or not monotonic.
1. $a_n = \frac{n-5}{n+5}$.
2. $a_n = \frac{\sqrt{n+5}}{7n+5}$.
3. $a_n = \frac{cos\ n}{5^n}$.
4. $a_n = \frac{1}{5n+7}$.

Solutions:
1. inc
This is because $a_n - a_{n-1} = \frac{n-5}{n+5} - \frac{n-6}{n+4} = \frac{n^2-n-20-n^2+n+30}{(n+5)(n+4)} = \frac{10}{(n+4)(n+5)} > 0$ for all $n > 0$.
2. dec
Let's take the inverse of a_n, which is $a_n^{-1} = \frac{7n+5}{\sqrt{n+5}} = \sqrt{n+5} + \frac{6n\sqrt{n+5}}{n+5}$. This is increasing and therefore sequence $\{a_n\}$ is decreasing.

3. not mono
This is because the component $cos\ n$ can be either > 0 or < 0.
4. dec
This is obvious because its inverse $5n + 7$ becomes a increasing sequence.