Homework 2

Problem 1

Prove that if G has order 1365 or 6545, then G is not simple.

Problem 2

Let G be a finite group and p a prime dividing the order of G. Let K be the set of all elements of G whose order is a power of p. Prove that K is a subgroup if and only if there exists a unique Sylow p-subgroup.

Problem 3

Let G be a group and let N be a normal subgroup of index n. Show that $g^n \in N$ for all $g \in G$.

Problem 4

Show that if G is a non-abelian finite group, then $|Z(G)| \leq 1/4|G|$.

Problem 5

Prove that the commutator subgroup of $SL_2(\mathbb{Z})$ is proper in $SL_2(\mathbb{Z})$.

Problem 6

(a) Find the centralizer in S_7 of $(123)(4567)$.

(b) How many elements of order 12 are there in S_7?

Problem 7

Prove that the symmetric group S_n is a maximal subgroup of S_{n+1}.

Problem 8

Let G be a group of order 16 with an element g of order 4. Prove that the subgroup of G generated by g^2 is normal in G.

Problem 9

We say that a group X is involved in a group G if X is isomorphic to H/K for some subgroups K, H of G with $K \trianglelefteq H$. Prove that if X is solvable and X is involved in the finite group G, then X is involved in a solvable subgroup of G.

Problem 10

Let G be a finite group and let N be a normal subgroup of G with the property that G/N is nilpotent. Prove that there exists a nilpotent subgroup H of G satisfying $G = HN$.

Problem 11

Let A be a commutative ring. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

(1) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$, where $r(\mathfrak{a})$ denotes the nil-radical of $\mathfrak{a}$.

(2) $V(0) = SpecA$, and $V(1) = \emptyset$.

(3) if $(E_i)_{i\in I}$ is any family of subsets of A, then $V(\cup_{i\in I} E_i) = \cap_{i\in I} V(E_i)$.

(4) $V(\mathfrak{a} \cup \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}$, $\mathfrak{b}$ of A.

Problem 12

Let A be a commutative ring. Prove that any $f = \sum_{i=0}^{\infty} a_i X^i \in A[[X]]$ is nilpotent, then a_i is nilpotent for any $a_i, i \geq 0$.

Problem 13

Let A be a commutative ring, and $\mathfrak{N}$ its nilradical. Show that the following are equivalent:

(1) A has exactly one prime ideal.

(2) every element of A is either a unit or nilpotent.

(3) $A/\mathfrak{R}$ is a field.

Problem 14

Let A be a ring (not necessarily commutative), n a positive integer, and $R = M_n(A)$, the ring of $n \times n$-matrices with coefficients in A. Let C denote the right A-module formed by column vectors of length n with coefficients in A.

(a) Show that the left action of R on C by formal matrix multiplication identifies R with $\mathrm{End}_A(C)$.

(b) For every A-submodule B of C, let I_B denote the set of all matrices $x \in R$ such that $xC \subset B$. Show that I_B consists of matrices x all of whose columns belong to B.

(c) Show that $B \mapsto I_B$ defines a bijection between the set of A-submodules B of C and the set of right ideals I_B of R.

1 Problem 15

Preserve the notations of problem 14 above.

(a) Establish a bijection between the set of two-sided ideals of A and the set of two-sided ideals of R.

(b) Prove that if A is a simple ring, show that R is also a simple ring.

Problem 16

Let R be a ring (not necessarily commutative).

(a) Suppose R is finite, satisfying $x \neq 0, y \neq 0 \Rightarrow xy \neq 0$. Show that R is a division ring if $R \neq (0)$.

(b) For any R, show that R is simple as a left R-module iff R is a division ring.