THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2022-2023 Term 1
Suggested Solutions of Homework Assignment 4

1. Evaluate the following integrals:

(a) $\displaystyle\int x\sec^2 x dx$

(b) $\displaystyle\int \sec^2 x \ln\tan x dx$

(c) $\displaystyle\int e^{-x}\sin 3x\, dx$

(d) $\displaystyle\int \frac{x^5}{(1+x^3)^3}dx$

(e) $\displaystyle\int x^2\ln\frac{1+x}{1-x}dx$

(f) $\displaystyle\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

Solution:

(a) $\displaystyle\int x\frac{1}{\cos^2 x} = \int x d\tan x = x\tan x - \int \tan x dx = x\tan x + \ln|\cos x| + C$

(b) $\displaystyle\int \sec^2 x\ln\tan x dx = \int \ln\tan x d\tan x = \tan x\ln\tan x - \tan x + C$

(c) $\displaystyle\int e^{-x}\sin 3x dx = -\int \sin 3x de^{-x} = -e^{-x}\sin 3x + 3\int e^{-x}\cos 3x dx$
on the other hand,
$\displaystyle\int e^{-x}\cos 3x dx = -\int \cos 3x de^{-x} = -e^{-x}\cos 3x - 3\int e^{-x}\sin 3x dx$
$\displaystyle\Rightarrow \int e^{-x}\sin 3x dx = -\frac{1}{10}(e^{-x}\sin 3x + 3e^{-x}\cos 3x) + C$

(d) $\displaystyle\int \frac{x^5}{(1+x^3)^3}dx = \frac{1}{3}\int \frac{x^3}{(1+x^3)^3}dx^3 = -\frac{1}{6}\int td\frac{1}{(1+t)^2} = -\frac{1}{6}(t\frac{1}{(1+t)^2} - \int \frac{1}{(1+t)^2}dt) = -\frac{1}{6}\frac{2x^3+1}{(1+x^3)^2} + C$

(e)

$$
\begin{aligned}
\int x^2\ln\frac{1+x}{1-x}dx &= \frac{1}{3}\int \ln\frac{1+x}{1-x}dx^3 \\
&= \frac{x^3}{3}\ln\frac{1+x}{1-x} - \frac{1}{3}\int x^3 d\ln\frac{1+x}{1-x} \\
&= \frac{x^3}{3}\ln\frac{1+x}{1-x} - \frac{2}{3}\int \frac{x^3}{1-x^2}dx \\
&= \frac{x^3}{3}\ln\frac{1+x}{1-x} - \frac{2}{3}\int \frac{x\left(x^2-1\right)+x}{1-x^2}dx \\
&= \frac{x^3}{3}\ln\frac{1+x}{1-x} + \frac{2}{3}\int xdx - \frac{2}{3}\int \frac{x}{1-x^2}dx \\
&= \frac{x^3}{3}\ln\frac{1+x}{1-x} + \frac{x^2}{3} + \frac{1}{3}\int \frac{1}{1-x^2}d\left(1-x^2\right) \\
&= \frac{x^3}{3}\ln\frac{1+x}{1-x} + \frac{x^2}{3} + \frac{1}{3}\ln\left(1-x^2\right) + C
\end{aligned}
$$

(f)

$$
\begin{aligned}
&\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx \\
=&\int \left(\frac{1+\sin x}{1+\cos x}\right) de^x \\
=&e^x \left(\frac{1+\sin x}{1+\cos x}\right) - \int e^x d\left(\frac{1+\sin x}{1+\cos x}\right) \\
=&e^x \left(\frac{1+\sin x}{1+\cos x}\right) - \int e^x \left(\frac{1+\sin x+\cos x}{(1+\cos x)^2}\right) dx \\
=&e^x \left(\frac{1+\sin x}{1+\cos x}\right) - \int e^x \left(\frac{1}{1+\cos x}\right) dx - \int e^x \left(\frac{\sin x}{(1+\cos x)^2}\right) dx \\
=&e^x \left(\frac{1+\sin x}{1+\cos x}\right) - \int e^x \left(\frac{1}{1+\cos x}\right) dx - \int e^x d\left(\frac{1}{1+\cos x}\right) \\
=&e^x \left(\frac{1+\sin x}{1+\cos x}\right) - \int e^x \left(\frac{1}{1+\cos x}\right) dx - e^x\left(\frac{1}{1+\cos x}\right) + \int e^x \left(\frac{1}{1+\cos x}\right) dx \\
=&e^x \left(\frac{\sin x}{1+\cos x}\right) + C
\end{aligned}
$$

2. Evaluate the following integrals:

(a) $\int \sin^4 x \, \cos^3 x dx$

(b) $\int \cos^4 x \, \sin^2 x dx$

(c) $\int \sec x \, \tan^3 x dx$

(d) $\int \sec^4 x \, \tan^6 x dx$

(e) $\int \frac{2e^x}{e^{2x}-4} dx$

(f) $\int \sec x dx$

(g) $\int \frac{1}{\sqrt{1-x^2}} dx$

(h) $\int \frac{1}{1+x^2} dx$

Solution:

(a)

$$
\begin{aligned}
\int \sin^4(x)\cos^3(x)dx &= \int \sin^4(x)\left(1-\sin^2(x)\right)\cos(x)dx \\
&= \int \sin^4(x)\cos(x)dx - \int \sin^6(x)\cos(x)dx \\
&= \int \frac{1}{5}\cdot 5\sin^4(x)\sin'(x)dx - \int \frac{1}{7}\cdot 7\sin^6(x)\sin'(x)dx \\
&= \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C, \text{ where } C \text{ is an arbitrary constant.}
\end{aligned}
$$

(b)

$$\begin{aligned}
&\int \cos^4(x)\sin^2(x)dx \\
=&\int \sin^2(x)\cos^2(x)\cdot\cos^2(x)dx \\
=&\int \frac{\sin^2(2x)}{4}\cdot\frac{\cos(2x)+1}{2}dx \\
=&\int \frac{\sin^2(2x)}{8}dx+\int \frac{\sin^2(2x)\cos(2x)}{8}dx \\
=&\int \frac{1-\cos(4x)}{16}dx+\int \frac{\sin^2(2x)\cdot 2\cos(2x)}{16}dx \\
=&\frac{x}{16}-\frac{\sin(4x)}{64}+\frac{\sin^3(2x)}{48}+C,\ \text{where } C \text{ is an arbitrary constant.}
\end{aligned}$$

(c)

$$\begin{aligned}
\int \sec(x)\tan^3(x)dx &= \int \tan^2(x)\sec'(x)dx \\
&= \int \left(\sec^2(x)-1\right)\sec'(x)dx \\
&= \frac{\sec^3(x)}{3}-\sec(x)+C,\ \text{where } C \text{ is an arbitrary constant.}
\end{aligned}$$

(d)

$$\begin{aligned}
&\int \sec^4(x)\tan^6(x)dx \\
=&\int \sec^2(x)\tan^6(x)\tan'(x)dx \\
=&\int \left(1+\tan^2(x)\right)\tan^6(x)\tan'(x)dx \\
=&\int \left(\tan^6(x)+\tan^8(x)\right)\tan'(x)dx \\
=&\frac{\tan^7(x)}{7}+\frac{\tan^9(x)}{9}+C,\ \text{where } C \text{ is an arbitrary constant.}
\end{aligned}$$

(e)

$$\begin{aligned}
\int \frac{2e^x dx}{e^{2x}-4} &= \int \frac{2du}{u^2-4} \\
&= \int \frac{1}{2}\left(\frac{1}{u-2}-\frac{1}{u+2}\right)du \\
&= \frac{1}{2}(\ln(|u-2|)-\ln(|u+2|))+C \\
&= \frac{1}{2}\ln\left(\left|\frac{u-2}{u+2}\right|\right)+C \\
&= \frac{1}{2}\ln\left(\left|\frac{e^x-2}{e^x+2}\right|\right)+C,\ \text{where } C \text{ is an arbitrary constant.}
\end{aligned}$$

We have used the substitution $u = e^x$.

(f)

$$\int \sec(x)dx$$
$$= \int \frac{dx}{\cos(x)} = \int \frac{\cos(x)dx}{\cos^2(x)} = \int \frac{\cos(x)dx}{1-\sin^2(x)}$$
$$= \int \frac{1}{2}\left(\frac{1}{1+\sin(x)} + \frac{1}{1-\sin(x)}\right)\sin'(x)dx$$
$$= \int \frac{1}{2}\cdot\frac{1}{1+\sin(x)}\cdot\frac{d}{dx}(1+\sin(x))dx + \int -\frac{1}{2}\cdot\frac{1}{1-\sin(x)}\cdot\frac{d}{dx}(1-\sin(x))dx$$
$$= \frac{1}{2}\ln(|1+\sin(x)|) - \frac{1}{2}\ln(|1-\sin(x)|) + C$$
$$= \frac{1}{2}\ln\left(\left|\frac{1+\sin(x)}{1-\sin(x)}\right|\right) + C$$
$$= \frac{1}{2}\ln\left(\left|\frac{(1+\sin(x))^2}{1-\sin^2(x)}\right|\right) + C$$
$$= \frac{1}{2}\ln\left(\left|\frac{(1+\sin(x))^2}{\cos^2(x)}\right|\right) + C$$
$$= \ln\left(\left|\frac{1+\sin(x)}{\cos(x)}\right|\right) + C$$
$$= \ln(|\sec(x)+\tan(x)|) + C, \text{ where } C \text{ is an arbitrary constant.}$$

(g) $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\sin'(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} = \int d\theta = \theta + C = \arcsin(x) + C$, where C is an arbitrary constant.
Here we have used the substitution $x = \sin(\theta)$.

(h) $\int \frac{dx}{1+x^2} = \int \frac{\sec^2(\theta)d\theta}{1+\tan^2(\theta)} = \int d\theta = \theta + C = \arctan(x) + C$, where C is an arbitrary constant.
Here we have used the substitution $x = \tan(\theta)$.

3. Evaluate the following integrals by trigonometric substitutions:

(a) $\int \frac{x^2dx}{(1-x^2)^{\frac{3}{2}}}$

(b) $\int \frac{dx}{\sqrt{4+x^2}}$

(c) $\int \sqrt{4-x^2}\,dx$

(d) $\int \frac{1}{(x^2+1)^2}\,dx$

Solution:

(a) Put $x = \sin t$, we have
$$\int \frac{x^2dx}{(1-x^2)^{\frac{3}{2}}} = \int \frac{\sin^2 t}{\cos^3 t}\cos t dt = \int \tan^2 t dt = \int (\sec^2 t - 1)dt$$
$$= \tan t - t + C = \frac{x}{\sqrt{1-x^2}} - \arcsin x + C.$$

(b) Put $x = 2\tan t, t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $\sqrt{4+x^2} = 2\sec t, dx = 2\sec^2 t dt$,

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{1}{2\sec t} 2\sec^2 t dt = \int \sec t dt = \int \frac{1}{\cos t} dt = \int \frac{\cos t}{\cos^2 t} dt$$

$$= \int \frac{1}{1-\sin^2 t} d(\sin t) = \frac{1}{2} \int \left[\frac{1}{1-\sin t} + \frac{1}{1+\sin t}\right] d\sin t$$

$$= \frac{1}{2} \ln \frac{|1+\sin t|}{|1-\sin t|} + C = \frac{1}{2} \ln \frac{|1+\sin t|^2}{\cos^2 x} + C = \ln|\sec t + \tan t| + C$$

$$= \ln|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}| + C.$$

(c) Put $x = 2\sin t$, then $dx = 2\cos t\, dt$

$$\int \sqrt{4 - 4\sin^2 t}\,(2\cos t\, dt) = \int 4\cos^2 t\, dt = \int 2(\cos(2t) + 1)\, dt$$

$$= \sin(2t) + 2t + C = \sin\left(2\arcsin(\frac{x}{2})\right) + 2\arcsin(\frac{x}{2}) + C$$

$$= \frac{1}{2} x\sqrt{4-x^2} + 2\arcsin(\frac{x}{2}) + C.$$

(d) Put $x = \tan t$, with $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $dx = \sec^2\ dt$

$$\int \frac{1}{(\tan^2 t + 1)^2} \sec^2 t\, dt = \int \frac{1}{\sec^2 t}\, dt = \frac{1}{2} \int (\cos(2t) + 1) dt$$

$$= \frac{1}{4}\sin(2t) + \frac{t}{2} + C = \frac{1}{2}\sin t\ \cos t + \frac{t}{2} + C = \frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x + C.$$

4. Prove the following reduction formulas.

(a) $I_n = \displaystyle\int \frac{x^n dx}{\sqrt{x+a}}$; $\quad I_n = \dfrac{2x^n\sqrt{x+a}}{2n+1} - \dfrac{2an}{2n+1} I_{n-1}$, $n \geq 1$.

(b) $I_n = \displaystyle\int \sin^n x dx$; $\quad I_n = -\dfrac{\cos x \sin^{n-1} x}{n} + \dfrac{n-1}{n} I_{n-2}$, $n \geq 2$.

(c) $I_n = \displaystyle\int \frac{1}{x^n(x+1)} dx$; $\quad I_n = \dfrac{1}{-n+1} x^{-n+1} - I_{n-1}$, $n \geq 2$.

(d) $I_n = \displaystyle\int_0^\pi x^n \sin x\, dx$; $\quad I_n = \pi^n - n(n-1)I_{n-2}$, $n \geq 2$, then find I_6.

Solution:

(a) $I_n = \int 2x^n d(\sqrt{x+a})$
$= 2x^n\sqrt{x+a} - \int 2nx^{n-1}\sqrt{x+a} dx$
$= 2x^n\sqrt{x+a} - \int \frac{2nx^{n-1}}{\sqrt{x+a}}(x+a)dx$
$= 2x^n\sqrt{x+a} - 2anI_{n-1} - \int \frac{2nx^n}{\sqrt{x+a}} dx$.
Therefore, $(2n+1)I_n = 2x^n\sqrt{x+a} - 2anI_{n-1}$, for $n \geq 1$.

(b) $I_n = \displaystyle\int \sin^n x dx = -\int \sin^{n-1} x d\cos x = -(\cos x \sin^{n-1} x - (n-1)\int \cos^2 x \sin^{n-2} dx) =$
$-\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \Rightarrow I_n = -\dfrac{\cos x \sin^{n-1} x}{n} + \dfrac{n-1}{n} I_{n-2}$

(c) For $n \geq 2$, $I_n = \displaystyle\int \frac{1}{x^n(x+1)}dx = \int \left(\frac{1+x}{x^n(x+1)} - \frac{x}{x^n(x+1)}\right) dx$
$= \displaystyle\int \frac{1}{x^n}dx - \int \frac{1}{x^{n-1}(x+1)}dx = \frac{1}{-n+1}x^{-n+1} - I_{n-1}.$

(d) $I_n = -\displaystyle\int_0^{\pi} x^n d(\cos x) = -x^n \cos x|_0^{\pi} + \int_0^{\pi} \cos x\, d(x^n) = \pi^n + \int_0^{\pi} \cos x\, n\, x^{n-1} dx$
$= \pi^n + n\displaystyle\int_0^{\pi} x^{n-1}\, d(\sin x) = \pi^n + n\left[\sin x\, x^{n-1}|_0^{\pi} - \int_0^{\pi} \sin x\,(n-1)x^{n-2}\, dx\right]$
$= \pi^n - n(n-1)I_{n-2} \Rightarrow I_n = \pi^n - n(n-1)I_{n-2}$, and $I_6 = \pi^6 - 30\pi^4 + 360\pi^2 - 1440$.

5. Find $F'(x)$ for the following functions:

(a) $F(x) = \displaystyle\int_{\pi}^{x} \frac{\cos y}{y}\, dy$

(b) $F(x) = \displaystyle\int_0^{x^3} e^{u^2}\, du$

(c) $F(x) = \displaystyle\int_x^{2x} (\ln t)^2\, dt$

(d) $F(x) = \displaystyle\int_{x^2}^{x^3} e^{\cos u}\, du$

(e) $F(x) = \displaystyle\int_1^{x} \frac{e^x + e^t}{t} dt$

(f) $F(x) = \displaystyle\int_{-\sqrt{\ln x}}^{\sqrt{\ln x}} \frac{\sin t}{t} dt$

Solution:

(a) $F'(x) = \dfrac{\cos x}{x}$

(b) $F'(x) = 3x^2 e^{x^6}$

(c) $F(x) = \displaystyle\int_1^{2x} (\ln t)^2 dt + \int_x^1 (\ln t)^2 dt \Rightarrow F'(x) = 2(\ln(2x))^2 - (\ln x)^2$

(d) $F'(x) = 3x^2 e^{\cos x^3} - 2x e^{\cos x^2}$

(e) $F(x) = e^x \displaystyle\int_1^x \frac{1}{t} dt + \int_1^x \frac{e^t}{t} dt \Rightarrow F'(x) = e^x(\ln x + \frac{2}{x})$

(f) $F(x) = \displaystyle\int_0^{\sqrt{\ln x}} \frac{\sin t}{t} dt - \int_0^{-\sqrt{\ln x}} \frac{\sin t}{t} dt = \frac{\sin\sqrt{\ln x}}{\sqrt{\ln x}} \frac{1}{2x\sqrt{\ln x}} - \frac{\sin(-\sqrt{\ln x})}{\sqrt{\ln x}} \frac{1}{2x\sqrt{\ln x}}$
$= \dfrac{\sin\sqrt{\ln x}}{x \ln x}$

6. Evaluate the following integrals of rational functions:

(a) $\displaystyle\int \frac{x^2}{1-x^2} dx$

(b) $\displaystyle\int \frac{4x+1}{x^2-6x+13} dx$

(c) $\displaystyle\int \frac{2x^3-x^2+3}{x^2-2x-3} dx$

(d) $\displaystyle\int \frac{x^2+1}{(x+1)^2(x-1)} dx$

(e) $\displaystyle\int \frac{2x^2-2}{2x^2-5x+2} dx$

(f) $\displaystyle\int \frac{-x+1}{2x^2+4x+5} dx$

Solution:

(a) $\int \frac{x^2 dx}{1-x^2} = \int \left(-1 + \frac{1}{1-x^2}\right) = -x + \frac{1}{2}\int \left(\frac{1}{1-x} + \frac{1}{1+x}\right)$
$= -x + \frac{-\ln|1-x| + \ln|1+x|}{2} + C$

(b) $\int \frac{4x+1}{x^2-6x+13}dx = \int \frac{4x-12+13}{x^2-6x+13}dx = 2\int \frac{2x-6}{x^2-6x+13}dx + 13\int \frac{1}{x^2-6x+13}dx$
$= 2\int \frac{d(x^2-6x+13)}{x^2-6x+13}dx + 13\int \frac{1}{(x-3)^2+4}dx$
$= 2\ln(x^2-6x+13) + 13\cdot\frac{2}{4}\int \frac{1}{(\frac{x-3}{2})^2+1}d(\frac{x-3}{2})$
$= 2\ln(x^2-6x+13) + \frac{13}{2}\arctan(\frac{x-3}{2}) + C$

(c) $\int \frac{2x^3-x^2+3}{x^2-2x-3}\,dx = \int (2x+3+12\frac{x+1}{(x-3)(x+1)})dx = x^2+3x+12\int \frac{1}{x-3}dx$
$= x^2+3x+12\ln|x-3| + C$

(d) $\int \frac{x^2+1}{(x+1)^2(x-1)},dx = \int \left[\frac{1}{2(x+1)} + \frac{1}{2(x-1)} - \frac{1}{(x+1)^2}\right]dx$
$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{1}{x+1} + C$

(e) $\int \frac{2x^2-2}{2x^2-5x+2}dx = \int \frac{2x^2-5x+2+5x-4}{(x-2)(2x-1)}dx = \int (1+\frac{5x-4}{(x-2)(2x-1)})dx =$
$\int (1+\frac{2(2x-1)+x-2}{(x-2)(2x-1)})dx = \int (1+\frac{2}{x-2}+\frac{1}{2x-1})dx$
$= x + 2\ln|x-2| + \frac{1}{2}\ln|2x-1| + C$

(f) $\int \frac{-x+1}{2x^2+4x+5}dx = \int \frac{-(x+1)+2}{2x^2+4x+5}dx$
$= -\frac{1}{4}\ln|2x^2+4x+5| + C + \int \frac{2}{2x^2+4x+5}dx$
$= -\frac{1}{4}\ln|2x^2+4x+5| + C + 2\int \frac{1}{2(x+1)^2+3}dx$
$= -\frac{1}{4}\ln|2x^2+4x+5| + C + \frac{2}{3}\int \frac{1}{(\frac{\sqrt{2}(1+x)}{\sqrt{3}})^2+1}dx$
$= -\frac{1}{4}\ln|2x^2+4x+5| + C + \frac{2}{3}\int \frac{\sqrt{\frac{3}{2}}}{(\frac{\sqrt{2}(1+x)}{\sqrt{3}})^2+1}d(\sqrt{\frac{2}{3}}(x+1))$
$= -\frac{1}{4}\ln|2x^2+4x+5| + C + \sqrt{\frac{2}{3}}\arctan(\sqrt{\frac{2}{3}}(x+1))$

7. Evaluate the following definite integrals:

(a) $\int_0^1 x^3\sqrt{1+3x^2}\,dx$

(b) $\int_0^\pi x\sin 2x\,dx$

(c) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

(d) $\displaystyle\int_0^5 |x^2-4x+3|\,dx$

(e) $\displaystyle\int_0^4 \ln(x^2+4)dx$

(f) $\displaystyle\int_0^{\frac{\pi}{3}} \tan^4(x)dx$

(g) $\displaystyle\int_0^1 \frac{x}{x^2+4x+5}dx$

(h) $\displaystyle\int_0^5 \frac{x}{\sqrt{9-x}}dx$

Solution:

(a) $$\int_0^1 x^3\sqrt{1+3x^2}\,dx = \int_0^1 \frac{1}{2}x^2\sqrt{1+3x^2}\,dx^2 = \frac{1}{2}\int_0^1 t\sqrt{1+3t}\,dt = \frac{1}{2}\int_0^1 t\,d\left(\frac{2}{9}(1+3t)^{\frac{3}{2}}\right)$$
$$= \frac{1}{9}\left[t(1+3t)^{\frac{3}{2}}\right]\Big|_0^1 - \frac{1}{9}\int_0^1 (1+3t)^{\frac{3}{2}}\,dt$$
$$= \frac{8}{9} - \frac{1}{9}\left[\frac{2}{15}(1+3t)^{\frac{5}{2}}\right]\Bigg|_0^1 = \frac{58}{135}$$

(b) $$\int_0^\pi x\sin 2x\,dx = \int_0^\pi -\frac{1}{2}x\,d\cos 2x = -\frac{1}{2}\left(\left[x\cos 2x\right]\Big|_0^\pi - \int_0^\pi \cos 2x\,dx\right) = -\frac{\pi}{2}$$

(c) $$\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = -\int_0^1 x\,d\sqrt{4-x^2} = -(x\sqrt{4-x^2}\Big|_0^1 - \int_0^1 \sqrt{4-x^2}dx)$$
$$= -\sqrt{3} + \int_0^1 \frac{4}{\sqrt{4-x^2}}\,dx - \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx,$$

so we have

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \frac{1}{2}\left(-\sqrt{3} + 4\int_0^1 \frac{1}{\sqrt{1-(x/2)^2}}d(x/2)\right)$$
$$= -\frac{\sqrt{3}}{2} + 2\arcsin(\frac{x}{2})\Big|_0^1 = -\frac{\sqrt{3}}{2} + \frac{\pi}{3}.$$

(d) $$\int_0^5 |x^2-4x+3|\,dx = \int_0^5 |(x-1)(x-3)|\,dx$$
$$= \int_0^1 \left(x^2-4x+3\right)\,dx - \int_1^3 \left(x^2-4x+3\right)\,dx + \int_3^5 \left(x^2-4x+3\right)\,dx$$
$$= \left(\frac{1}{3}-2+3\right)\times 2 - \left(\frac{1}{3}\cdot 3^3 - 2\cdot 3^2 + 3\cdot 3\right)\times 2 + \left(\frac{1}{3}\cdot 5^3 - 2\cdot 5^2 + 3\cdot 5\right)$$
$$= \frac{28}{3}$$

(e) $$\int_0^4 \ln\left(x^2+4\right) dx = \int_0^4 \ln\left(x^2+4\right)\cdot 1\, dx$$
$$= x\ln\left(x^2+4\right)\Big|_0^4 - \int_0^4 \frac{2x^2}{x^2+4}dx$$
$$= x\ln\left(x^2+4\right)\Big|_0^4 - \int_0^4 \left(2-\frac{8}{x^2+4}\right)dx$$
$$= x\ln\left(x^2+4\right)\Big|_0^4 - \int_0^4 2dx + \int_0^4 4\cdot\frac{1/2}{1+(x/2)^2}dx$$
$$= x\ln\left(x^2+4\right)\Big|_0^4 - 2x\Big|_0^4 + 4\arctan\left(\frac{x}{2}\right)\Big|_0^4$$
$$= 4\ln(20) - 8 + 4\arctan(2)$$

(f) $$\int_0^{\frac{\pi}{3}} \tan^4(x)dx = \int_0^{\frac{\pi}{3}} \tan^2(x)\left(\sec^2(x)-1\right)dx$$
$$= \int_0^{\frac{\pi}{3}} \left(\tan^2(x)\sec^2(x) - \tan^2(x)\right)dx$$
$$= \int_0^{\frac{\pi}{3}} \left(\tan^2(x)\sec^2(x) - \sec^2(x)+1\right)dx$$
$$= \left(\frac{\tan^3(x)}{3} - \tan(x) + x\right)\Bigg|_0^{\frac{\pi}{3}}$$
$$= \frac{(\sqrt{3})^3}{3} - \sqrt{3} + \frac{\pi}{3} = \frac{\pi}{3}.$$

(g) $$\int_0^1 \frac{x}{x^2+4x+5}dx = \int_0^1 \frac{(x+2)}{(x+2)^2+1}dx - \int_0^1 \frac{2}{(x+2)^2+1}dx$$
$$= \left(\frac{1}{2}\ln\left[(x+2)^2+1\right]\right)\Bigg|_0^1 - \left(2\arctan(x+2)\right)\Big|_0^1$$
$$= \frac{1}{2}\ln(10) - \frac{1}{2}\ln(5) - 2\arctan(3) + 2\arctan(2)$$
$$= \frac{1}{2}\ln(2) - 2\arctan(3) + 2\arctan(2)$$

(h) $$\int_0^5 \frac{x}{\sqrt{9-x}}dx = -\int_9^4 \frac{9-u}{\sqrt{u}}du = \int_4^9 \left(9u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right)du$$
$$= \left(18u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_4^9 = 18\times(9^{\frac{1}{2}} - 4^{\frac{1}{2}}) - \frac{2}{3}\times(9^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \frac{16}{3}$$

8. Evaluate the following indefinite integrals by using the t-substitution:

(a) $$\int \frac{1}{3+2\sin x+\cos x}dx$$

(b) $$\int \frac{1}{2+\cos x}dx$$

Solution: We use the substitution $t = \tan\left(\frac{x}{2}\right)$ below. Then $\sin(x) = \dfrac{2t}{1+t^2}$,

$\cos(x) = \dfrac{1-t^2}{1+t^2}$ and $dx = \dfrac{2}{1+t^2}\,dt$, and so we have the following results.

(a)

$$\begin{aligned}
&\int \frac{1}{3+2\sin(x)+\cos(x)}\,dx \\
=&\int \frac{1}{3+2\cdot\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt \\
=&\int \frac{2}{2t^2+4t+4}\,dt = \int \frac{1}{t^2+2t+2}\,dt \\
=&\int \frac{1}{1+(t+1)^2}\,dt \\
=&\arctan(t+1)+C \\
=&\arctan\left(\tan\left(\frac{x}{2}\right)+1\right)+C, \text{ where } C \text{ is an arbitrary constant.}
\end{aligned}$$

(b)

$$\begin{aligned}
&\int \frac{1}{2+\cos(x)}\,dx \\
=&\int \frac{1}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt \\
=&\int \frac{2}{3+t^2}\,dt = \int \frac{2}{\sqrt{3}}\cdot\frac{1/\sqrt{3}}{1+(t/\sqrt{3})^2}\,dt \\
=&\frac{2}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)+C \\
=&\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right)+C, \text{ where } C \text{ is an arbitrary constant.}
\end{aligned}$$

9. Suppose that $f : [-1,1] \to (0,\infty)$ is an even, continuous function such that

- $\displaystyle\int_{-1}^{1} f(x)\,dx = 1$;
- $f(x)$ is strictly increasing over $[-1,0]$.

(a) Show that $f(x)$ attains its global maximum at 0.

(b) Let

$$G(r) = \int_{-r}^{r} f(x)\,dx.$$

Use first principles to show that $G(r)$ is differentiable over $[-1,1]$ and find its derivative.

Solution:

(a) Since $f(x)$ is an even function on $[-1,1]$, and it is strictly increasing over $[-1,0]$, we know that $f(x)$ is strictly decreasing over $[0,1]$. Hence, $f(x)$ attains its global maximum at $x=0$.

(b) Since $f(x)$ is even on $[-1,1]$,

$$f(-x) = f(x), \text{ for all } x \in [-1,1].$$

We can get the result that, for all $r \in [-1,1]$,

$$\int_{-r}^{0} f(x)\,dx = \int_{0}^{r} f(x)\,dx. \text{ (Substitute } -x \text{ for } x).$$

So we have

$$G(r) = \int_{-r}^{r} f(x)\,dx = 2\int_{0}^{r} f(x)\,dx.$$

Hence, by the Fundamental Theorem of the Calculus, we know that $G(r)$ is differentiable on $[-1,1]$, since $f(x)$ is continuous on $[-1,1]$. And

$$G^{'}(r) = 2f(r), \text{ for all } r \in [-1,1].$$

10. By considering a suitable integral, evaluate the following limits:

(a) $\lim_{n\to\infty} \sum_{k=0}^{n-1} \ln\left(\frac{n+k}{n}\right)^{\frac{1}{n}}$

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n^2+k^2}{n^3+k^3}$

Solution:

(a)
$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=0}^{n-1} \ln\left(\frac{n+k}{n}\right)^{\frac{1}{n}} &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n}\ln\left(1+\frac{k}{n}\right) \\
&= \int_0^1 \ln(1+x)dx \\
&= [x\ln(1+x)]\Big|_0^1 - \int_0^1 \frac{x}{1+x}dx \text{ (Integration by part)} \\
&= [x\ln(1+x)]\Big|_0^1 - \int_0^1 \left(1-\frac{1}{1+x}\right)dx \\
&= [x\ln(1+x)]\Big|_0^1 - (x - \ln|1+x|)\Big|_0^1 \\
&= \ln 2 - (1-\ln 2) \\
&= 2\ln 2 - 1
\end{aligned}$$

(b) $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n^2+k^2}{n^3+k^3}$$

$$\begin{aligned}
&=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+\left(\frac{k}{n}\right)^2}{1+\left(\frac{k}{n}\right)^3}\cdot\frac{1}{n}\\
&=\int_0^1\frac{1+x^2}{1+x^3}dx\\
&=\int_0^1\frac{1+x^2}{(x+1)\left(x^2-x+1\right)}dx\\
&=\int_0^1\left(\frac{2}{3}\cdot\frac{1}{x+1}+\frac{1}{3}\cdot\frac{x+1}{x^2-x+1}\right)dx\\
&=\frac{2}{3}\int_0^1\frac{1}{x+1}dx+\frac{1}{3}\int_0^1\frac{x+1}{x^2-x+1}dx\\
&=\frac{2}{3}\int_0^1\frac{1}{x+1}dx+\frac{1}{6}\int_0^1\frac{(2x-1)+3}{x^2-x+1}dx\\
&=\frac{2}{3}\int_0^1\frac{1}{x+1}dx+\frac{1}{6}\int_0^1\frac{2x-1}{x^2-x+1}dx+\frac{1}{2}\int_0^1\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx\\
&=\frac{2}{3}\big(\ln|x+1|\big)\Big|_0^1+\frac{1}{6}\big(\ln\left|x^2-x+1\right|\big)\Big|_0^1+\frac{1}{2}\left(\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}\right)\Bigg|_0^1\\
&=\frac{2}{3}\ln 2+\frac{\pi}{3\sqrt{3}}
\end{aligned}$$

11. Let f_0 be a continuous function on $\mathbb{R}$. For each positive integer n, define the function f_n on $\mathbb{R}$ by $f_n(x)=\displaystyle\int_0^x f_{n-1}(t)dt$ for any $x\in\mathbb{R}$.

(a) Show that whenever m, n are positive integers,

$$\int_0^x (x-t)^{m-1}f_n(t)dt=\frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)dt$$

for any $x\in\mathbb{R}$.

(b) Show that whenever n is a positive integer, $f_n(x)=\dfrac{1}{(n-1)!}\displaystyle\int_0^x (x-t)^{n-1}f_0(t)dt$ for any $x\in\mathbb{R}$.

Solution:

(a) Note that, for any positive integer n, $f_n(0)=0$ and

$$f_n'(x)=\frac{d}{dx}\int_0^x f_{n-1}(t)dt=f_{n-1}(x)\qquad\text{for all } x\in\mathbb{R},$$

by the Fundamental Theorem of Calculus.

Hence, for any $x \in \mathbb{R}$, integration by parts yields

$$\begin{aligned}
&\int_0^x (x-t)^{m-1} f_n(t)dt \\
&= \int_0^x -\frac{1}{m} f_n(t) \frac{d}{dt}((x-t)^m)\, dt \\
&= \left[-\frac{1}{m}(x-t)^m \cdot f_n(t)\right]_0^x - \int_0^x -\frac{1}{m}(x-t)^m \cdot f_n'(t)\, dt \\
&= \left[-\frac{1}{m}(x-x)^m \cdot f_n(x)\right] - \left[-\frac{1}{m}(x-0)^m \cdot f_n(0)\right] - \int_0^x -\frac{1}{m}(x-t)^m \cdot f_{n-1}(t)\, dt \\
&= \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)\, dt.
\end{aligned}$$

(b) Let n be a positive integer and let $x \in \mathbb{R}$. By (a), we have

$$\begin{aligned}
f_n(x) &= \int_0^x f_{n-1}(t)dt \\
&= \int_0^x (x-t)^0 f_{n-1}(t)dt \\
&= \frac{1}{1}\int_0^x (x-t)^1 f_{n-2}(t)dt \\
&= \frac{1}{1\cdot 2}\int_0^x (x-t)^2 f_{n-3}(t)dt \\
&= \cdots \\
&= \frac{1}{k!}\int_0^x (x-t)^k f_{n-k-1}(t)dt \\
&= \cdots \\
&= \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f_0(t)dt.
\end{aligned}$$