THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2022-2023 Term 1
Homework Assignment 3
Due Date: 24 November 2022

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ______________________ Date ______________________

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests there as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to 2022R1 University Mathematics (MATH1010ABCDEF)
  2. Choose Tools in the left-hand column
  3. Scroll down to the bottom of the page
  4. The green Gradescope icon will be there

- Late assignments will receive a grade of 0.

- For the declaration sheet:

  Either

  Print out the cover sheet (i.e. the first page of this document), and sign and date the statement of Academic Honesty. Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction).
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. By using Lagrange's mean value theorem, or otherwise, show that

   (a) $\sin x \leq x$ for all $x \in [0, +\infty)$.

   (b) $(1+x)^p \geq 1 + px$ for any $p \geq 1$ and $x \geq 0$.

2. Let $0 < a < b < \frac{\pi}{2}$. Prove that there exists $a < \xi < b$ such that

$$\ln\left(\frac{\cos a}{\cos b}\right) = (b-a)\tan\xi.$$

3. Show that for all $0 < a < b \leq 1$,

$$(b-a)(1+\ln a) < \ln\left(\frac{b^b}{a^a}\right) < (b-a)(1+\ln b).$$

4. Evaluate the following limits.

   (a) $\lim_{x\to 0} \dfrac{\sin^{-1} x - \tan^{-1} x}{x^3}$

   (b) $\lim_{x\to 0} \log_{\tan x}(\tan 2x)$

   (c) $\lim_{x\to 0^+} \tan x \ln \sin x$

   (d) $\lim_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{x}{x-1}\right)$

   (e) $\lim_{x\to +\infty} \dfrac{e^{1+\ln x}}{\ln(1+e^x)}$

5. Evaluate the following limits.

   (a) $\lim_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

   (b) $\lim_{x\to 1} x^{\frac{2x}{x-1}}$

   (c) $\lim_{x\to 0} \dfrac{(1+x)^x - 1}{x^2}$

   (d) $\lim_{x\to +\infty} \left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^x$

6. For each of the following functions $f(x)$, find

   - domain of f and x, y-intercepts
   - asymptotes of $y = f(x)$
   - $f'(x)$, local maximum, local minimum, intervals on which f is increasing, decreasing
   - $f'(x)$, points of inflection, intervals on which f is concave up, down

   Then sketch the graph of $y = f(x)$.

   (a) $f(x) = \dfrac{x}{(x-2)^2}$

   (b) $f(x) = \dfrac{x^2+5x+7}{x+2}$

   (c) $f(x) = \dfrac{x^2}{x^2-2x+2}$

   (d) $f(x) = x^{\frac{2}{3}} - 1$

7. Find the Taylor series up to the term in $(x-c)^3$ of the functions about $x = c$.

(a) $\dfrac{1}{1+x}$; $c = 1$.

(b) $\dfrac{2-x}{3+x}$; $c = 1$.

(c) $\dfrac{x}{(x-1)(x-2)}$; $c = 0$.

(d) $\cos x$; $c = \frac{\pi}{4}$.

(e) $\sin^2 x$; $c = 0$

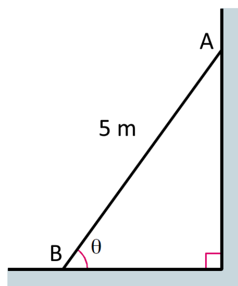
(f) $\ln x$; $c = e$.

(g) 3^x; $c = 0$.

(h) $\sqrt{2+x}$; $c = 1$.

(i) $\dfrac{1}{\sqrt{7-3x}}$; $c = 1$.

8. In the following figure



a ladder with length 5 m leans against a wall. The point of contact A between the ladder and the wall slides down at a constant speed of 0.8 m/s. When A is 4.8 m above the ground,

(a) find the sliding speed of B away from the wall

(b) find the rate of change of θ (in degree/s, correct to 2 decimal places)

9. Sketch a graph of a twice-differentiable function $f : (0, \infty) \to \mathbb{R}$ which satisfies the followings:

- $f(1) = 5$ and $f(2) = 3$
- $\lim_{x \to 0^+} f(x) = -\infty$ (DNE) and $\lim_{x \to \infty} f(x) = 1$
- $f'(x) > 0$ over $(0, 1)$ and $f'(x) < 0$ over $(1, \infty)$
- $f''(x) < 0$ over $(0, 2)$ and $f''(x) > 0$ over $(2, \infty)$

On your graph, label any local maximum(s), local maximum(s), point of inflection(s) and asymptote(s) (if any).

10. Find the global maximum and minimum (if exist) of

$$f(x) = x^{\frac{4}{5}} e^{-x}$$

with domain $[-1, \infty)$

11. Suppose

$$f(x) = \sqrt{1+x}$$

(a) Find the Taylor polynomials, $T_n(x)$, of order $n = 0, 1, 2, 3$ of $f(x)$ with center 0.

(b) Use $T_0(x), T_1(x), T_2(x), T_3(x)$ to approximate the value of $\sqrt{1.2}$

12. Find the exact value of
$$\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$