THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2022-2023 Term 1
Homework Assignment 2
Due Date: October 24, 2022 (Monday)

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

____________________ ____________________
Signature Date

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests here as well. For submitting your PDF homework on Gradescope, here are a few tips.
- Late assignments will receive a grade of 0.
- For the declaration sheet:

  Either

  Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

  Or

  Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."
- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction).
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. The function f is continuous at $x = 0$ and is defined for $-1 < x < 1$ by

$$f(x) = \begin{cases} \dfrac{2a}{x}(e^x - 1) & \text{if } -1 < x < 0 \\ 1 & \text{if } x = 0 \\ \dfrac{bx\cos x}{1-\sqrt{1-x}} & \text{if } 0 < x < 1. \end{cases}$$

Determine the values of the constants a and b.

2. Determine whether the following functions are differentiable at $x = 0$.

(a) $f(x) = \begin{cases} 1 + 3x - x^2, & \text{when } x < 0 \\ x^2 + 3x + 2, & \text{when } x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

(c) $f(x) = \sin|x|$

(d) $f(x) = x|x|$

3. Let $f(x) = |x|^3$.

(a) Find $f'(x)$ for $x \neq 0$.

(b) Show that $f(x)$ is differentiable at $x = 0$.

(c) Determine whether $f'(x)$ is differentiable at $x = 0$.

4. Let

$$f(x) = \begin{cases} (x-2)^2 \sin\left(\dfrac{1}{x-2}\right), & \text{when } x \neq 2; \\ 0, & \text{when } x = 2. \end{cases}$$

(a) Is f continuous on $\mathbb{R}$?

(b) Is f differentiable on $\mathbb{R}$?

(c) Is f' continuous on $\mathbb{R}$?

5. Find natural domains of the following functions and differentiate them on their natural domains. You are not required to do so from first principles.

(a) $f(x) = \dfrac{1 + \sin x}{1 + 2\cos x}$.

(b) $f(x) = (1 + \tan^2 x)\cos^2 x$.

(c) $f(x) = \ln(\ln(\ln x))$

(d) $f(x) = \ln|\sin x|$

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose f is differentiable at $x = 0$, with $f'(0) = a$. Show that $f(x) = ax$.

7. Find $\dfrac{dy}{dx}$ if

   (a) $x^2 + y^2 = e^{xy}$
   (b) $x^3 y + \sin(xy^2) = 4$
   (c) $y = \tan^{-1}\sqrt{x}$
   (d) $y = 3^{\sin x}$
   (e) $y = x^{\ln x}$
   (f) $y = x^{x^x}$
   (g) $y = \sin(\cos(\sin x))$

8. Find $\dfrac{d^2y}{dx^2}$ if

   (a) $y = \ln\tan x$
   (b) $y = \sin^{-1}\sqrt{1-x^2}$
   (c) $y^2 = x^3 - x$
   (d) $\cos^2 y + \sin x = 1$

9. Find the n-th derivative of the following functions for all positive integers n.

   (a) $f(x) = (e^x + e^{-x})^2$, $x \in \mathbb{R}$
   (b) $f(x) = \dfrac{1}{1-x^2}$, $x \in (-1, 1)$
   (c) $f(x) = \sin x \cos x$, $x \in \mathbb{R}$
   (d) $f(x) = \cos^2 x$, $x \in \mathbb{R}$
   (e) $f(x) = \dfrac{x^2}{e^x}$, $x \in \mathbb{R}$

10. Find all points (x_0, y_0) on the graph of

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 8$$

where lines tangent to the graph at (x_0, y_0) have slope -1.

11. The chain rule says

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

or equivalently,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $y = f(u)$ and $u = g(x)$.

(a) Give examples to show

$$(f \circ g)''(x) \neq f''(g(x)) \cdot g''(x),$$

or equivalently,

$$\frac{d^2y}{dx^2} \neq \frac{d^2y}{du^2} \cdot \frac{d^2u}{dx^2},$$

where $\frac{d^2y}{dx^2}$ denotes the second derivative of $y = f(x)$.

(b) Prove

$$(f \circ g)''(x) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).$$

12. (a) Suppose $a, b > 0$ are constants, and

$$y = \frac{1}{ab} \arctan\left(\frac{b}{a} \tan x\right)$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Express $\frac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

(b) Suppose $a, b > 0$ are constants, and

$$y = \ln\left|\frac{a + b\tan x}{a - b\tan x}\right|$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left\{\pm\arctan\left(\frac{a}{b}\right)\right\}$. Express $\frac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

13. Let a, b be real numbers and $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{3 + a\sqrt{1 - \frac{2}{3}x}}{x} & \text{if } x < 0 \\ 1 + b\tan\left(\dfrac{x}{10}\right) & \text{if } x \geq 0. \end{cases}$$

Assume that $f(x)$ is continuous at $x = 0$.

(a) Determine the value of a and justify your computation.

(b) If we further assume that f is differentiable at 0, determine the value of b and justify your answer.