# Week 9
# Partial Derivatives
# Chain Rule

## Differentiability

If a function $f(x, y)$ in two variables is differentiable at $(a, b)$, then geometrically it means that there exists a tangent plane to the graph $z = f(x, y)$ of $f$ at the point $(a, b, f(a, b))$ on the graph. An equation which describes this tangent plane is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

or equivalently:

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + (z - f(a, b)) = 0.$$

Hence, this is the plane in $\mathbb{R}^3$ which contains the point $(a, b, f(a, b))$, and has $\vec{n} = \langle -f_x(a, b), -f_y(a, b), 1 \rangle$ as a normal vector.

---

Let $f(x, y)$ be a function in $2$ variables. The existence of $f_x(a, b)$ and $f_y(a, b)$ does ***not*** guarantee the differentiability of $f$ at $(a, b)$. However,

> **Theorem.**
>
> If $f_x$ and $f_y$ are continuous on an open region containing $(a, b)$, then $f$ is differentiable at $(a, b)$.

---

> **Theorem.**
>
> If $f$ is differentiable at $P$, then it is continuous at $P$.

---

## Higher Order Partial Derivatives

Since, $\frac{\partial f}{\partial x_i}$ is itself a function in $n$ variables, we can consider its partial derivative with respect to any of the variables $x_j$. And we can further consider partial derivatives of *that* partial derivative, and so on. The notation is as follows:

$$\frac{\partial^2 f}{\partial x_i^2} = f_{x_i x_i} := \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right).$$

For $j \neq i$,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j} := \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right).$$

For $m \in \mathbb{N}$,

$$\frac{\partial^m f}{\partial x_i^m} = f_{\underbrace{x_i x_i \cdots x_i}_{m \text{ times}}} := \frac{\partial}{\partial x_i}\left(\frac{\partial^{m-1} f}{\partial x_i^{m-1}}\right)$$

For $i_1, i_2, \ldots, i_m \in \{1, 2, 3, \ldots, n\}$,

$$\frac{\partial^m f}{\partial x_{i_m} \partial x_{i_{m-1}} \partial x_{i_{m-2}} \cdots \partial x_{i_1}} = f_{x_{i_1} x_{i_2} \cdots x_{i_m}} := \frac{\partial}{\partial x_{i_m}}\left(\frac{\partial^{m-1} f}{\partial x_{i_{m-1}} \partial x_{i_{m-2}} \cdots \partial x_{i_1}}\right).$$

---

**Theorem.**

Let $x$ and $y$ be two of the variables of a function $f$. If $f_{xy}$ and $f_{yx}$ are continuous on an open region containing a point $P$, then:

$$f_{xy}(P) = f_{yx}(P).$$

---

## Chain Rule

**Theorem.** If $f(x_1, x_2, \ldots, x_n)$ is a differentiable function in $n$ variables, and each $x_i = x_i(s_1, s_2, \ldots, s_m)$ $(i = 1, 2, \ldots, n)$ is a differentiable function in $m$ variables, then $f$ is differentiable as a function in $s_1, s_2, \ldots, s_m$, with:

$$\frac{\partial f}{\partial s_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial s_i}.$$

**Definition.**

The **gradient** of $f(x_1, x_2, \ldots, x_n)$ at $P \in \mathbb{R}^n$ is:

$$\nabla f(P) = \langle f_{x_1}(P), f_{x_2}(P), \ldots, f_{x_n}(P) \rangle \in \mathbb{R}^n$$

---

Hence,

$$\frac{\partial f}{\partial s_i} = \nabla f \cdot \frac{\partial \vec{x}}{\partial s_i},$$

where:

$$\frac{\partial \vec{x}}{\partial s_i} := \left\langle \frac{\partial x_1}{\partial s_i}, \frac{\partial x_2}{\partial s_i}, \ldots, \frac{\partial x_n}{\partial s_i} \right\rangle.$$

---