Week 8
Limits, Continuity
Partial Derivatives

Limits and Continuity

Definition.

Let f be a real-valued function in n variables with domain D. For $P_0 \in \mathbb{R}^n$, $L \in \mathbb{R}$, we say that the limit of f at P_0 is L, written:

$$\lim_{P \to P_0} f(P) = L,$$

if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(P) - L| < \varepsilon$ whenever $0 < \left|\overline{P_0 P}\right| < \delta$, $P \in D$. (Here, $\left|\overline{P_0 P}\right|$ is the distance between P_0 and P.)

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Put differently, the limit of f at P_0 is L if: for all $\varepsilon > 0$, there exists an open ball $B_\delta(P_0)$ with radius $\delta > 0$, centred at P_0, such that the values of f at all $P \neq P_0$ in $B_\delta(P_0) \cap D$ are within ε of L. Note that P_0 does not have to be in D for the limit $\lim\limits_{P_0 \to P} f(P)$ to exist.

Example.

Find:

- $$\lim_{(x,y)\to(0,1)} \frac{x^2 + y}{4y - x}$$
- $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Theorem.

Sandwich Theorem If we have:

$$g(x, y) \leq f(x, y) \leq h(x, y)$$

for all $(x, y) \neq (a, b)$ in an open disk centered at (a, b), and:

$$\lim_{(x,y)\to(a,b)} g(x, y) = \lim_{(x,y)\to(a,b)} h(x, y) = L,$$

then:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

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Example.

Find:

$$\lim_{(x,y)\to(0,0)} x\sin\left(\frac{1}{x^2+y^2}\right).$$

Theorem.

Let $\gamma,\psi:\mathbb{R}\longrightarrow\mathbb{R}^n$ be the parameterization of two paths in $\mathbb{R}^n$, with $\gamma(0)=\psi(0)=P_0$. If $\lim\limits_{t\to 0} f(\gamma(t))$ or $\lim\limits_{t\to 0} f(\psi(t))$ does not exist, or the two limits are not equal to each other, then the limit $\lim\limits_{P\to P_0} f(P)$ does not exist.

Example.

Consider $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, where:

$$f(x,y)=\frac{xy}{x^2+y^2}.$$

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Let:

$$\gamma(t)=(t,t),\quad t\in\mathbb{R},$$
$$\psi(t)=(t,-t),\quad t\in\mathbb{R}.$$

Then,

$$\gamma(0)=\psi(0)=(0,0),$$

and:

$$\lim_{t\to 0} f(\gamma(t))=\lim_{t\to 0}\frac{t\cdot t}{t^2+t^2}=\lim_{t\to 0}\frac{t^2}{2t^2}=\frac{1}{2},$$

$$\lim_{t\to 0} f(\psi(t))=\lim_{t\to 0}\frac{t\cdot(-t)}{t^2+(-t)^2}=\lim_{t\to 0}-\frac{t^2}{2t^2}=-\frac{1}{2},$$

Since $\lim\limits_{t\to 0} f(\gamma(t))\neq\lim\limits_{t\to 0} f(\psi(t))$, we conclude that the limit $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy}{x^2+y^2}$ does not exist.

Remark.

Let $P_0 = (x_0, y_0)$. If $\lim_{x \to x_0} f(x, y_0) = \lim_{x \to y_0} f(x_0, y) = L$, it is not necessarily true that $\lim\limits_{(x,y) \to (x_0,y_0)} f(x, y) = L$, or that the limit even exists.

Definition.

We say that a function f in n variables is **continuous** at $P_0 \in \text{Domain}(f)$ if:

$$\lim_{P \to P_0} f(P) = f(P_0).$$

Every elementary function (a function constructed from constants, power functions, trigonometric, inverse trigonometric, exponential and logarithmic functions, via addition, subtraction, multiplication, division and composition) is continuous at all points in its domain.

Example.

- Every polynomial in n variables (e.g. $f(x, y, z) = x^2yz + 5yz^2 + 16y^3 - 8$) is continuous everywhere.
- Every rational function in n variables is continuous at all points where the function is defined.
- $f(x, y) = e^{\cos(x^2+y^2)}$ is continuous at all $(x, y) \in \mathbb{R}^2$.
- $f(x, y) = \dfrac{1}{\sqrt{x^2 + y}}$ is continuous at all $(x, y) \in \mathbb{R}^2$ such that $x^2 + y > 0$.

Partial Derivatives

Definition.

Let f be a function in n variables, $P = (a_1, a_2, \ldots, a_n) \in \mathrm{Domain}(f)$. For $i = 1, 2, \ldots, n$, we define the **partial derivative** with respect to x_i of f at P to be:

$$\begin{aligned}\frac{\partial f}{\partial x_i}(P) &= \left(\frac{d}{dx_i} f(a_1, a_2, \ldots, a_{i-1}, \underbrace{x_i}_{i\text{-th coordinate}}, a_{i+1}, \ldots, a_n)\right)\Bigg|_{x_i = a_i} \\ &= \lim_{h \to 0} \frac{f(a_1, a_2, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(P)}{h}\end{aligned}$$

As P varies, $\dfrac{\partial f}{\partial x_i}$ is itself a function in n variables, with:

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}$$

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Notation. Another notation for $\dfrac{\partial f}{\partial x_i}$ is f_{x_i}.

Example.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for:

- $$f(x, y) = x^5 y^3 + e^y \sin(x^2 y)$$
- $$f(x, y, z) = x^2 y + y z^3 + \frac{x}{y}$$

Example.

Find $\frac{\partial f}{\partial x}\Big|_{(x,y)=(0,0)}$ for

$$f(x, y) = \begin{cases} 0, & y = -x; \\ (x^2 + y^2) \sin(\frac{1}{x+y}), & y \neq -x. \end{cases}$$

Differentiability

Definition. A real-valued function f in two variables is differentiable at $(a, b) \in \mathrm{Domain}(f)$ if:

- $f_x(a, b)$ and $f_y(a, b)$ exist;
- There exists an open ball $B \subseteq \mathrm{Domain}(f)$, centered at (a, b), such that for all $(x, y) \in B$, we have:

$$f(x, y) - f(a, b) = f_x(a, b) \underbrace{\Delta x}_{(x-a)} + f_y(a, b) \underbrace{\Delta y}_{(y-b)} + \varepsilon_x \Delta x + \varepsilon_y \Delta y$$

for some $\varepsilon_x, \varepsilon_y$ which approach 0 as $\Delta x, \Delta y$ approach 0.

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In general, a function $f(x_1, x_2, \ldots, x_n)$ in n variables is differentiable at $P_0 = (a_1, a_2, \ldots, a_n) \in \text{Domain}(f)$ if $f_{x_i}(P_0)$ exists for $i = 1, 2, \ldots, n$, and for $P = (x_1, x_2, \ldots, x_n)$ inside an open ball centered at P_0, we have

$$f(P) = f(P_0) + \sum_{i=1}^{n} f_{x_i}(P_0)(x_i - a_i) + \varepsilon(P)\,|P_0 P|$$

for some real-valued $\varepsilon(P)$ such that $\lim_{P \to P_0} \varepsilon(P) = 0$. Here, $|P_0 P|$ denotes the distance between P_0 and P.

If a function $f(x, y)$ in two variables is differentiable at (a, b), then geometrically it means that there exists a tangent plane to the graph $z = f(x, y)$ of f at the point $(a, b, f(a, b))$ on the graph. An equation which describes this tangent plane is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

or equivalently:

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + (z - f(a, b)) = 0.$$

Hence, this is the plane in $\mathbb{R}^3$ which contains the point $(a, b, f(a, b))$, and has $\vec{n} = \langle -f_x(a, b), -f_y(a, b), 1 \rangle$ as a normal vector.
