Functions in n Variables

Given $n \in \mathbb{N}$. A real-valued function f in n variables is a map:

$$f : D \longrightarrow \mathbb{R},$$

where the domain D is a subset of $\mathbb{R}^n$.

> **Example.**
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> $$f : \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 0\} \longrightarrow \mathbb{R}$$
>
> $$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}.$$

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If the domain D of f is not explicitly given, it is assumed to be the natural domain (or: maximal domain, domain of definition) of f. Namely, it is the set of all points in $\mathbb{R}^n$ on which the expression defining f is well-defined.

Regions in $\mathbb{R}^n$

Let D be a region/subset of $\mathbb{R}^n$. A point $P_0 \in \mathbb{R}^n$ is said to be an **interior point** of D if there exists an open disc/ball $B_r(P_0)$ of nonzero radius r, centered at P_0, such that $B_r(P_0) \subseteq D$. Here,

$$B_r(P_0) = \left\{ P \in \mathbb{R}^n \ : \ \left| \overrightarrow{P_0 P} \right| < r \right\}.$$

In particular, an interior point of D must lie in D. A point $P_0 \in \mathbb{R}^n$ is said to be a **boundary point** of D if, for all $r > 0$, the open ball $B_r(x_0, y_0)$ has nonempty intersection with both D and $\mathbb{R}^n \backslash D := \{P \in \mathbb{R}^n : P \notin D\}$ The **boundary** ∂D of a region D is the set consisting of its boundary points. The **interior** of a region D is the set consisting of its interiors points. A region D is said to be **open** if every point of D is an interior point. It is said to be **closed** if it contains its boundary.

Example.

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The region:

$$D = \{(x, y) \mid x^2 + y^2 < 4\} \subseteq \mathbb{R}^2$$

is open, with boundary:

$$\partial D = \{(x, y) \mid x^2 + y^2 = 4\}$$

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The region:

$$D = \{(x, y) \mid y \leq x^2\} \subseteq \mathbb{R}^2$$

is closed, for it contains its boundary:

$$\partial D = \{(x, y) \mid y = x^2\}$$

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Let:

$$D = \{(x, y) \mid -1 \leq y < 1\} \subseteq \mathbb{R}^2$$

The boundary of D is:

$$\partial D = \{(x, y) \mid y = \pm 1\}.$$

Since some points of the boundary (e.g. $(x, y) = (5, -1)$) belong to D, but others (e.g. $(x, y) = (4, 1)$) do not, the region D is neither open nor closed.

For a function in two variables, $f : D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^2$, the **graph** of f is the set of points $(x, y, z) \in \mathbb{R}^{2+1} = \mathbb{R}^3$ such that:

$$z = f(x, y), \quad (x, y) \in D.$$

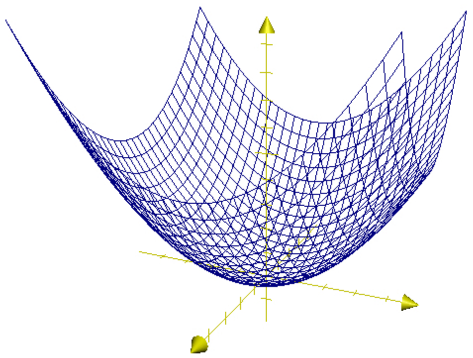
(We often write $z = f(x, y)$ to denote the graph of a function f in two variables.)

Example.

The following figure shows the graph of:

$$f : [-1, 1] \times [-1, 1] \longrightarrow \mathbb{R}$$

$$f(x, y) = x^2 + y^2$$



Level Sets

Definition.

For a function $f : D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^n$, in n variables, and $c \in \mathbb{R}$, the **level set** of f corresponding to c is the set of points $(x_1, x_2, \ldots, x_n) \in D$ such that

$$f(x_1, x_2, \ldots, x_n) = c$$

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Remark.

If $n = 2$, then a level set of f is typically a curve in the xy-plane, and is often called a **level curve**.
If $n = 3$, then a level set is typically a surface in the xyz-space, and is often called a **level surface**.

Example.

$f(x,y) = x^2 + y^2$.

- For $c = -2, -1$, the level sets $f(x,y) = x^2 + y^2 = c$ are empty.
- For $c = 0$, the level set $f(x,y) = x^2 + y^2 = 0$ consists of the single point $(0,0)$.
- For $c > 0$, the level set $f(x,y) = x^2 + y^2 = c$ is the circle in $\mathbb{R}^2$ centred at the origin with radius $\sqrt{c}$.

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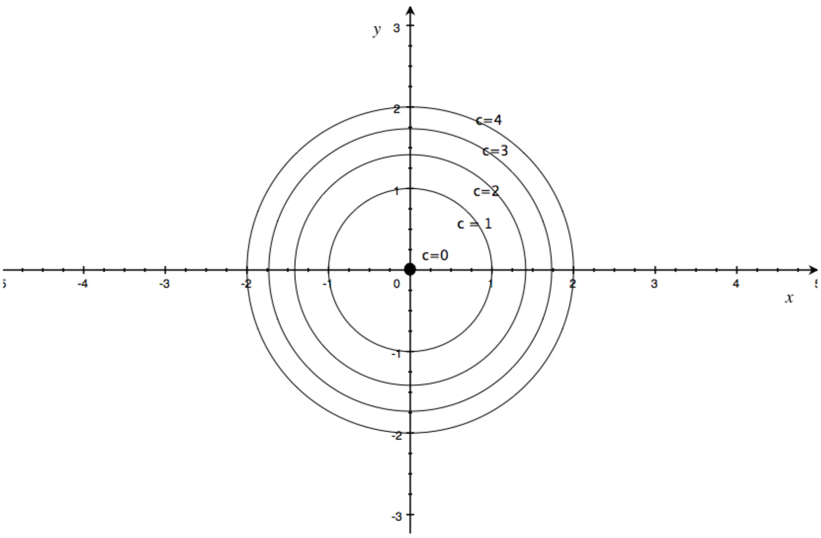

Each level set $f(x,y) = c$ corresponds to (the projection onto the xy-plane of) the intersection of the surface $z = f(x,y)$ and the horizontal (hence ``level'') plane $z = c$:

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